WHAT DO *WE* MEAN BY MATHEMATICAL PROOF?

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Introduction

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INITIAL THOUGHTS

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- "We," the worldwide community of mathematicians, or
- "We," the students in your classroom.

The answer to what *we* mean by mathematical proof may change depending on the context of the word "we."

We'll talk about this question in detail, but first let's look at some conjectures and try to prove them together.

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As we go, keep in mind the question

"What do *we* mean by mathematical proof?"

and any other related questions that arise from your discussion.

Two Conjectures

Conjecture #1: The spelling of every whole number shares at least one letter with the spelling of the next whole number.

That is,

- "one" and "two" share the letter 'o'
- "two" and "three" share the letter 't'
- "three" and "four" share the letter 'r'
- … and so on (?)

Is this true for every whole number?

Source: Meyer (2016).

Conjecture #2: You have 3^n coins, all of which are identical except for one counterfeit, which appears identical to the others, but is very slightly heavier. You have a balance scale with which you can weigh coins against each other in order to find the counterfeit, but you may only use it n times.

- Given 3 coins, one of which is counterfeit: you must identify the counterfeit using the balance scale only once.
- Given 9 coins, one of which is counterfeit: you must identify the counterfeit using the balance scale only twice.
- … and so on.

Is is always possible to identify the counterfeit from 3^n coins using the balance scale n times?

Source: Harel (2001).

Discussing these Conjectures

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Are there any edge cases worth considering?

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This conjecture is not infinite! The largest number I know a standard name for is “centillion”: 10^{303}.

CONJECTURE 2

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This is a perfect question to introduce *mathematical induction*.

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Formal and Practical Meanings of Proof

One possible starting point for a definition is this:

> *"Everybody knows what a mathematical proof is. A proof of a mathematical theorem is a sequence of steps which leads to the desired conclusion. The rules to be followed by such a sequence of steps were made explicit when logic was formalized early in this century, and they have not changed since."*

(Rota, 1997, p. 183)

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However, as Rota (1997) points out later in the same article, this description is *unrealistic* in describing the proofs that mathematicians actually write.

This formal notion of proof was held by mathematicians at the beginning of the 20th century, including those of the “formalist” school of David Hilbert.

The formalists sought to create mathematics free of contradictions by writing their statements in a formal language, and proving them using formal rules of inference. (Snapper, 1979)

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This purely formal notion of proof does not reflect the *practice* of mathematicians.

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When a mathematician reads a proof to determine its validity, he or she makes a judgment based on whether or not it is convincing: in other words, based on *subjective* criteria.

Davis and Hersh (1981) make this point in an imaginary dialogue between "The Ideal Mathematician" and a philosophy student who asks for a definition of proof:

> ***I.M.:*** *Well, this whole thing was cleared up by the logician Tarski, I guess, and some others, maybe Russell or Peano. Anyhow, what you do is, you write down the axioms of your theory in a formal language,...Then you show that you can transform the hypothesis step by step, using the rules of logic, till you get a conclusion. That's a proof.*

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> ***Student:*** *Really? That's amazing!...I've never seen that done before.*

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Student: *But even that doesn't sound like what was done in my courses and textbooks...Then what really is a proof?*

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I.M.: *Well, it's an argument that convinces someone who knows the subject.*
Student: *Someone who knows the subject? Then the definition of proof is subjective; it depends on particular persons...*
I.M.: *No, no. There's nothing subjective about it! Everybody knows what a proof is...*

The practical notion of proof does not do away with *rigor*. Detlefsen (2008) argues that rigor and “formalizability” are independent notions. Proofs, as written by mathematicians, are presented in such a way as to make their rigor clear to the reader, but do not require formalization.

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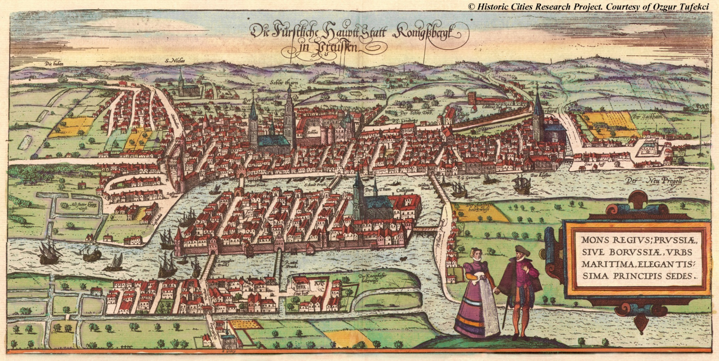
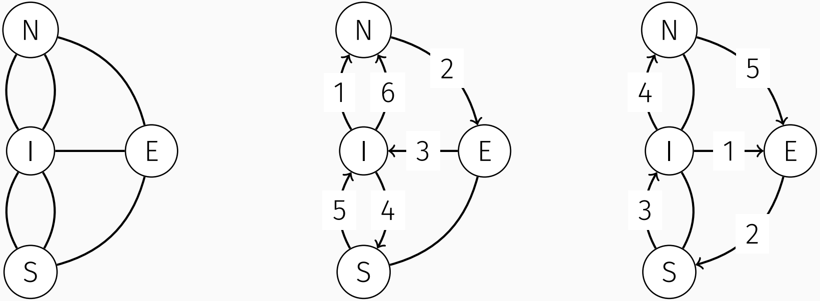
In other words, rigor itself is a set of standards agreed upon by the community of mathematicians, and that set of standards does not include strict formalization.

The Bridges of Königsberg puzzle is a famous example of a recreational mathematics problem that spawns new mathematics.

Challenge: Can you cross over every bridge exactly once and return to your starting point?



Abstraction, and some attempts at a solution:



Theorem: If it is possible to cross over every bridge exactly once, then every section of town has an even number of bridges connecting to it.

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Proof 1: Pick any section of town. Then, for every time we enter this section, we also have to leave it. Therefore, every time we pass through this section of town, we must use two bridges. Therefore, no matter how many times we pass through, the number of bridges connected to that section must be must be even.

Theorem: In a connected graph, if there exists a closed trail using every edge exactly once, then every vertex has even degree.

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Proof 2: Suppose there exists a closed trail using every edge exactly once. Then, that closed trail contributes 2 to the degree of each vertex each time that vertex appears in the trail. Therefore, it contributes an even number to the degree of each vertex. Since every edge is included exactly once in this trail, the total degree of every vertex is accounted for by this trail, and the total degree is even.

Roles of Proof

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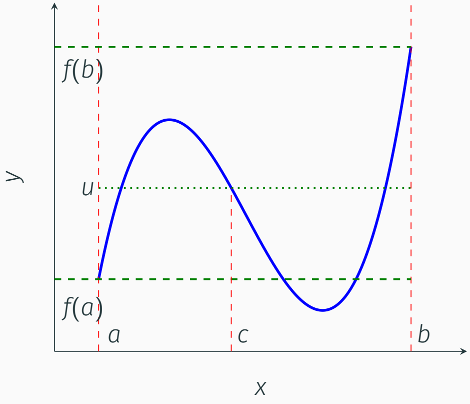
- verification,
- explanation,
- systematization,
- discovery,
- intellectual challenge,
- and communication.

To research mathematicians, the primary role of proof is that of *verification*.

In the mathematics classroom, however, the primary role is of *explanation*.

Intermediate Value Theorem: Let f be continuous on $[a, b]$ and assume $f(a) < f(b)$. Then for every u such that $f(a) < u < f(b)$, there exists a $c \in [a, b]$ such that $f(c) = u$.

Proof:



Intermediate Value Theorem: Let f be continuous on $[a, b]$ and assume $f(a) < f(b)$. Then for every u such that $f(a) < u < f(b)$, there exists a $c \in [a, b]$ such that $f(c) = u$.

Proof: Let S be the set of all $x \in [a, b]$ such that $f(x) \leq u$. Then S is non-empty, since $a \in S$. S is bounded above by b, therefore, the supremum $c = \sup S$ exists.

We want to show that $f(c) = u$:

Let $\epsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. This means that $f(x) - \epsilon < f(c) < f(x) + \epsilon$ for all $x \in (c - \delta, c + \delta)$.

Since c is the supremum of S, there exists $v \in (c - \delta, c]$ contained in S. Therefore, $f(c) < f(v) + \epsilon \leq u + \epsilon$. Also, there exists $w \in (c, c + \delta)$ not contained in S, so we have $f(c) > f(w) - \epsilon > u - \epsilon$.

Putting these inequalities together, we have that $u - \epsilon < f(c) < u + \epsilon$ for any $\epsilon > 0$, and therefore $f(c) = u$.

Learning to Write Proofs

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The current mathematics curriculum in the United States is heavily weighted toward a formal meaning of proof. High school geometry classes emphasize highly formalized "two-column" proofs.

For students who do not major in mathematics in college, this is very likely the only experience they will ever have with mathematical proof.

As students progress through a college mathematics degree, they begin to see the meaning of proof evolve - first in calculus, where they may see $\delta - \epsilon$ proofs of limits, or proofs of the Fundamental Theorem of Calculus, and beyond in a introduction-to-proof class and later upper-division classes.

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What we mean by mathematical proof evolves as we make progress towards “thinking like a mathematician.”

The Need for Mathematical Proofs

Why do we need to prove things?

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"If mathematical proof is the aspirin, how do you create the headache?"

This is a phrase Dan Meyer uses to describe *intellectual need*.

One problem I like to use to create intellectual need is this:

$$(1+2+3+4)^2 = 10^2 = 100$$

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$$1^3+2^3+3^3+4^3 = 1+8+27+64 = 100$$

In fact,

$$\left(\sum_{k=1}^{n} k\right)^2 = \sum_{k=1}^{n} k^3.$$

This fact is just weird enough to surprise students into needing a proof!

Another headache-causing question:

> *A salt solution is 99% water. Some of the water evaporates, leaving a solution that is 98% water. How much of the original solution remains after evaporation?*

(Adapted from Usiskin et al, 2003)

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Once you have solved it, can you *prove* your answer is correct?

Final Thoughts

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FINAL THOUGHTS

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Complicating the matter is the fact that we use mathematical proof in different, but related roles.

Because we are unable to precisely define what a proof is, we should not be surprised that students often fail to meet our implicit and evolving standards of proof.

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Thank you!

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For the full paper on which this talk is based, see:

CadwalladerOlsker, T. (2011) What Do We Mean By Mathematical Proof? *Journal of Humanistic Mathematics* 1(1) p. 33-60; https://scholarship.claremont.edu/jhm/vol1/iss1/4/