Where Do Kepler's Laws Hold?

CMC^3 Recreational Math Conference

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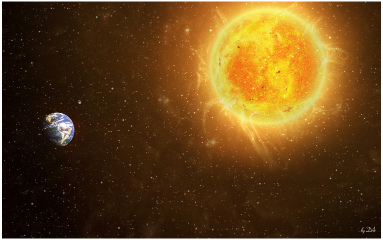
The classical Kepler problem

The Kepler Problem

Determine the motion of a planet around a sun.



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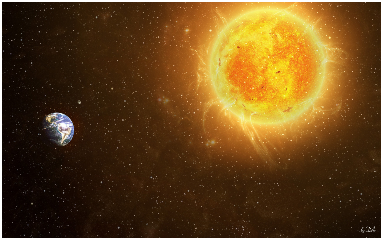

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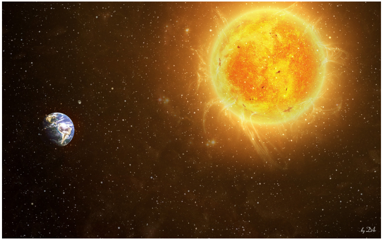

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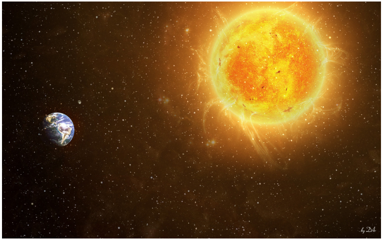

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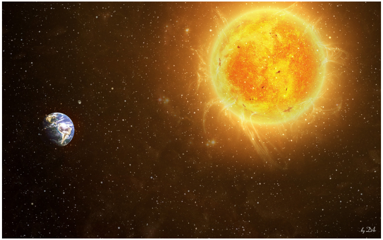

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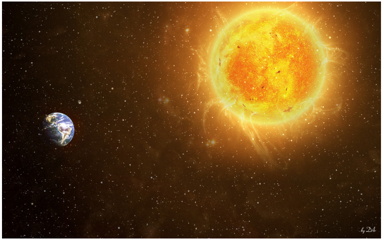

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- $r = \sqrt{x^2 + y^2 + z^2}$, c = constant

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Problem (Hamiltonian formalism)

Integrate the flow of the Hamiltonian $H : \mathbb{R}^6 \to \mathbb{R}$ where

- $H = K + U =$ total energy
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$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{x}},$$

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Note: U is the fundamental solution to the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

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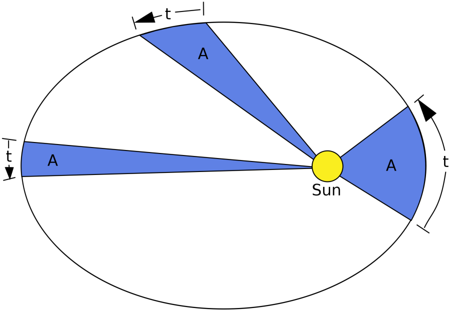

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1609: First and Second Laws. 1619: Third Law.

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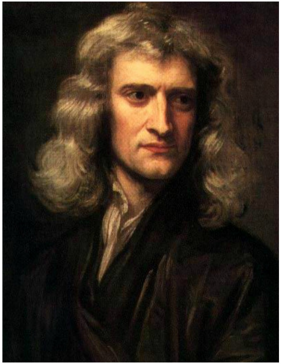

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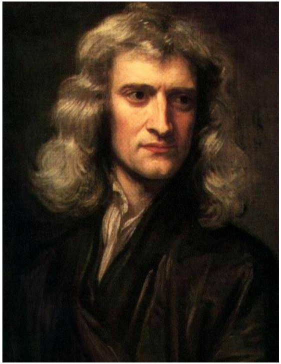

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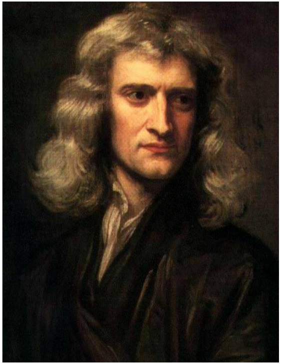

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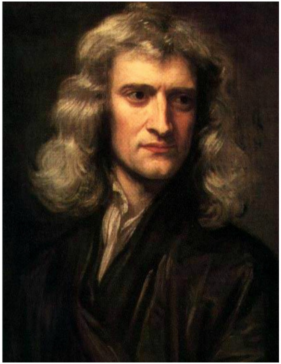

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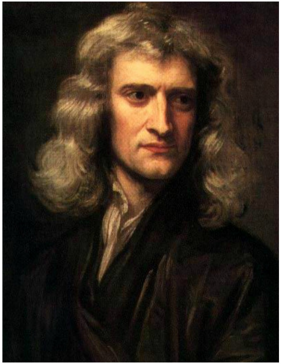

Proved (mathematically) Kepler's Laws from Newton's Laws (including law of gravitation).

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- Implies planetary orbits lie in a plane. Thus, the Kepler problem in $\mathbb{R}^3$ easily reduces to $\mathbb{R}^2$.
- Partially explains the observation that the sun, moon, and all planets follow the same path in the sky, called the *ecliptic*.

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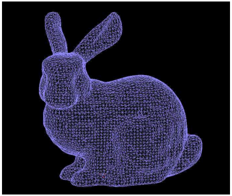

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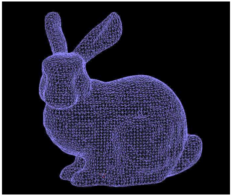

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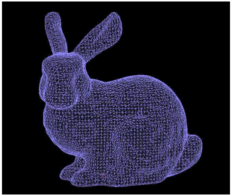

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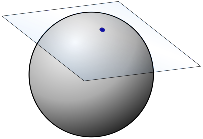

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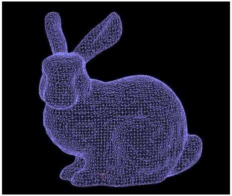

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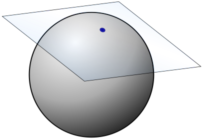

- A *Riemannian structure* is an inner product $\langle \cdot, \cdot \rangle$ at each tangent space.

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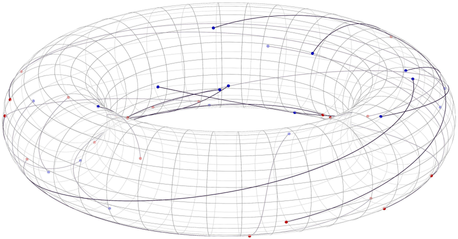

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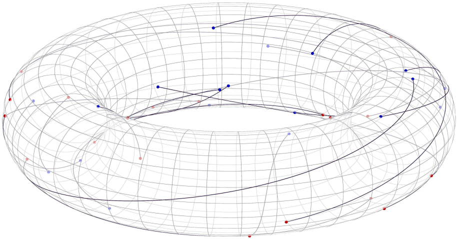

- So, *a Riemannian manifold is a smooth space where we can measure distances.*

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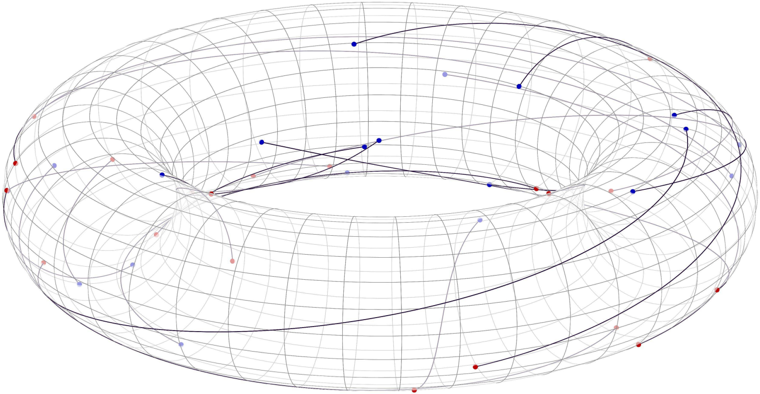

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- Try to integrate flow of Hamiltonian

$$H = K + U.$$

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- 2011-now, Montgomery, Shanbrom: Heisenberg 3-space $\mathcal{H}$, posed, partially solved

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- For the curved Third Law to hold, we need a homogeneous potential on a metric manifold *with dilations*.

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- A: Only Euclidean spaces!

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But where do we go...?

The Kepler problem on the Heisenberg group

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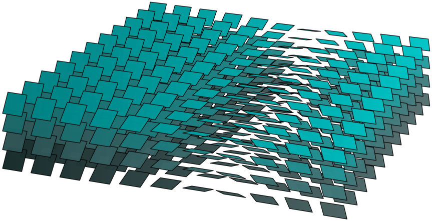

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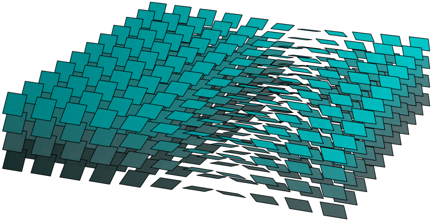

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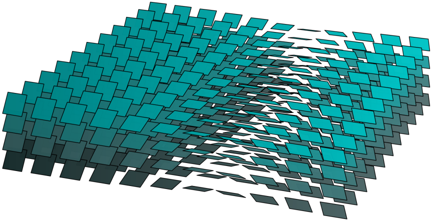

- So can think of a field of subspaces, and require all curves tangent to these subspaces.
- In other words, certain directions of travel are forbidden.

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Definition

The Heisenberg group is the Lie group $\mathcal{H}$ homeomorphic to $\mathbb{R}^3$, with group law

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \tfrac{1}{2}(x_1 y_2 - x_2 y_1)).$$

The distribution

- Three vector fields give a basis for each tangent space:

$$X = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}$$

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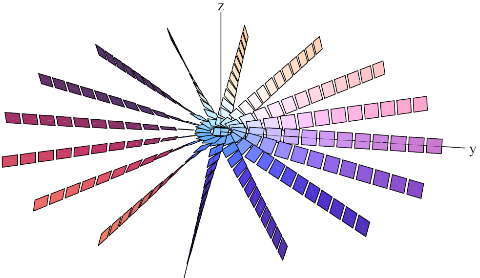

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$$l(\gamma) = \int_a^b \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle} dt$$

- Induces a distance function (metric) in the usual way

$$d(p, q) = \inf l(\gamma),$$

where the infimum is taken over all *horizontal* curves connecting p to q.

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Heisenberg geodesics are helices.

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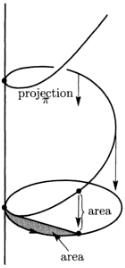

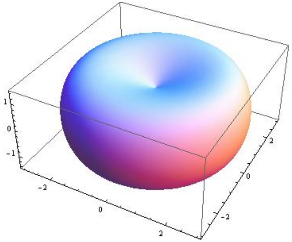

Figure: Left: The z-coordinate must grow like the area traced out. Right: The Heisenberg unit sphere.

The Kepler Problem on $\mathbb{R}^3$

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- Potential $U = -\frac{c}{r}$ with $c = \frac{1}{4\pi}$ is the fundamental solution to the $\mathbb{R}^3$ Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

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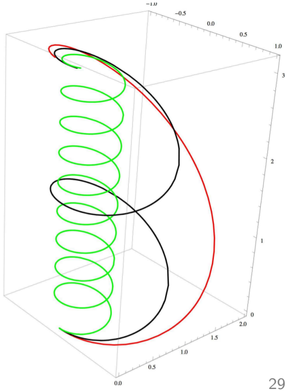

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This will be our potential energy!

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Problem

Integrate the flow of the Hamiltonian

$$\begin{aligned} H &= K + U \\ &= \tfrac{1}{2}(P_X^2 + P_Y^2) - \tfrac{2}{\pi}\left((x^2 + y^2)^2 + \tfrac{1}{16}z^2\right)^{-1/2} \end{aligned}$$

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Hamilton's equations

Equations of motion

$$\dot{x} = P_X$$
$$\dot{y} = P_Y$$
$$\dot{z} = \frac{1}{2}xP_Y - \frac{1}{2}yP_X$$

$$\dot{p}_x = -\frac{1}{2}P_Y p_z - \frac{4}{\pi}x(x^2+y^2)\left((x^2+y^2)^2 + \tfrac{1}{16}z^2\right)^{-3/2}$$
$$\dot{p}_y = \frac{1}{2}P_X p_z - \frac{4}{\pi}y(x^2+y^2)\left((x^2+y^2)^2 + \tfrac{1}{16}z^2\right)^{-3/2}$$
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Conic sections, just like Kepler in $\mathbb{R}^3, \mathbb{S}^3$, and $\mathbb{H}^3$!

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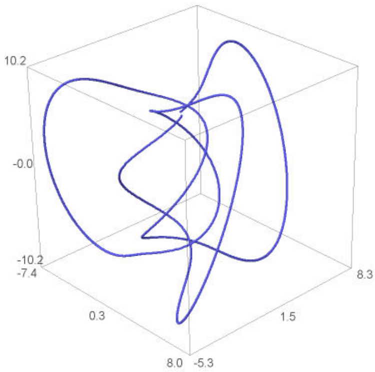
Proof gives *existence* only, but can use numerics to find.

A periodic orbit; $k = 3$



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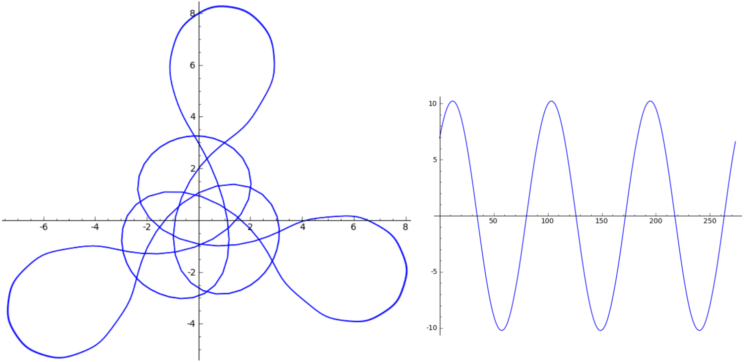
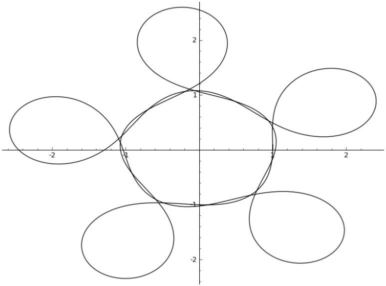
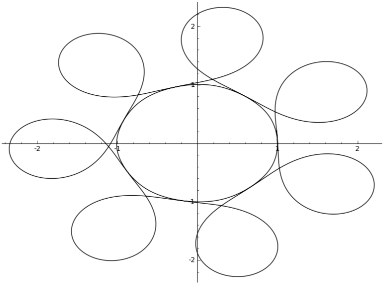


Figure: Left: Projection to *xy*-plane. Right: *z*-coordinate over time; grows like area traced out by projection

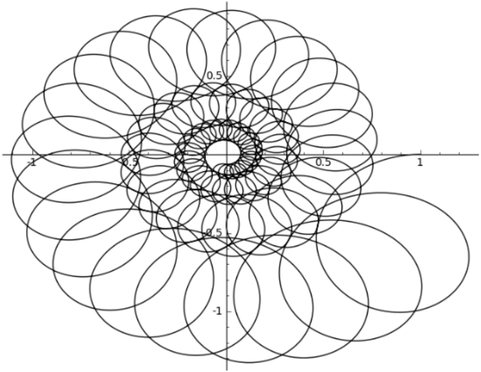
Periodic orbits, $k = 5$ and 7





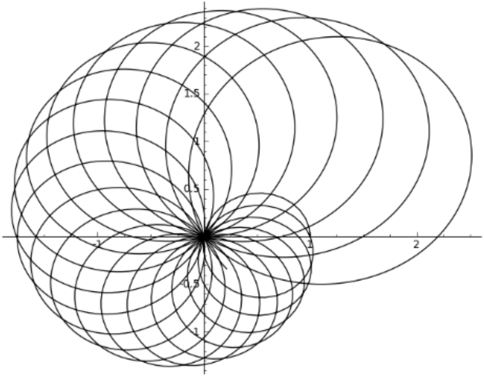
Other numerical solutions

We can numerically approximate some other orbits...



Other numerical solutions

...which clearly reflect the helical geometry.



Kepler-Heisenberg Laws

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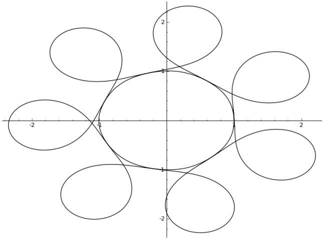

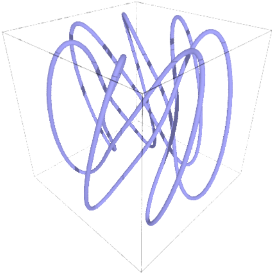

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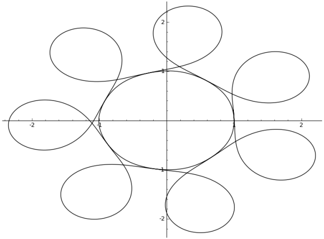

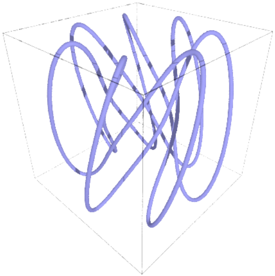

- The Second Law does hold; it is equivalent to the conservation of angular momentum, and the system is rotationally symmetric (Noether's theorem)

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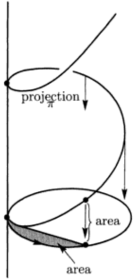

Summary

Table: The Kepler Problem in different geometries

Geometry	$\exists$ periodic orbits	1st law	2nd law	3rd law
$\mathbb{R}^3$	✓	✓	✓	✓
$\mathbb{S}^3$	✓	✓	✓	×
$\mathbb{H}^3$	✓	✓	✓	×
$\mathcal{H}$	✓	×	✓	✓

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- What happens in higher (odd) dimensional Heisenberg groups?

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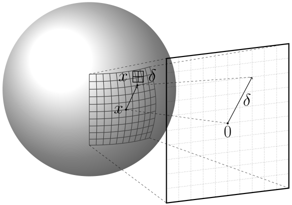

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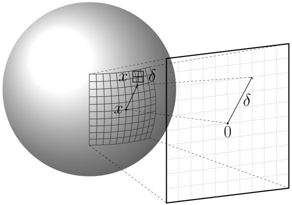

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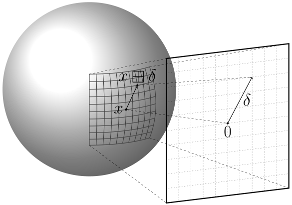

- Unlike the earth, which we can now photograph from space, we cannot step outside the universe and look down from above to observe its global geometry.
- However, just as ancient scientists deduced the shape of the earth from local observations, we could conceivably determine the shape of the universe from small scale data.

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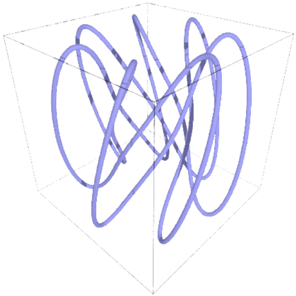
So if we some day find a star orbiting a black hole along this path,



Conclusions

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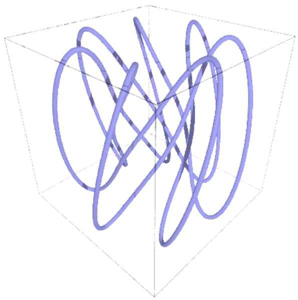

we may find ourselves living in a Heisenbergly-curved universe.

Thank you for listening!

SACRAMENTO STATE
Department of Mathematics & Statistics

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