McDonald's Happy Meal Toys

The Real Cost Of Trying To Collect Them All

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CMC^3 Conference
April 18, 2015

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View of Las Vegas at night

View of Las Vegas at night

My family.

Introduction


TRANSFORMERS
ROBOTS IN DISGUISE
McPlay Power
Back
Scan your toy! Unlock a game!


My Little Pony
Equestria Girls
McPlay Power
Back
Scan your toy! Unlock a game!

This problem arose because my son wanted to collect all 8 toys!

Two Questions I Started To Ask Myself

1) How Many Happy Meals Do I Have To Buy Before I Collect All 8 Toys?

2) How Much Money Will I Spend Until I Have Collected All 8 Toys?

These Questions Can Be Answered By Applying the Coupon Collector Problem!

What Is The Coupon Collector Problem?

Suppose an urn contains r different balls, and the balls are drawn without replacement until k types of balls have been drawn at least m times each. Let n equal the number of balls drawn.

The Coupon Collector Problem deals with the number of balls drawn until k different balls have been drawn.

Also known as the occupancy problem, random allocation problem, and the birthday problem

The Classic Coupon Collector Problem

Coupons Are Collected One At A Time

The Types of Coupons Are Known In Advance

The Probability Of Collecting A Coupon Of Any Type Is Equally Likely

The Waiting Time Is The Number Of Coupons We Need To Collect Until We Have A Complete Set

Think About The Question Another Way:

How Many Times Do You Need To Flip A Fair Coin Until You See Both Sides At Least Once?

How Many Times Do You Need To Roll A Fair Die Until You See All Six Faces At Least Once?

Let's Try It:

Flip The Coin Until You Have Seen Both Faces At Least Once. How Many Times Did You Flip The Coin?

Roll The Fair Die Until You Have Seen All Six Faces At Least Once. How Many Times Did You Roll The Die?

We Want To Find The Expected Waiting Time For Experiments Of This Type!!

Remember, The Probability Of Obtaining A Coupon Of Any Type Is Equally Likely

From Elementary Probability Theory, If There Are k Coupons To Collect, Then The Expected Waiting Time Is:

$$\text{E[WT]} = k\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}\right) \qquad (1)$$

This Can Also Be Approximated By:

$$\text{E[WT]} = k\left(\log k+\kappa\right) \qquad (2)$$

Where $\kappa = 0.57721566$ is Euler's Constant

If $k = 2$:

By Equation 1, E[WT] = 3

By Equation 2, E[WT] = 2.5407

If $k = 6$:

By Equation 1, E[WT] = 14.7

By Equation 2, E[WT] = 14.2139

In This Set Of Happy Meal Toys, There Were 8 Toys.

Hence, Using Equation 1, We Get:

$E[WT] = 21.7429$

Using Equation 2, We Get:

$E[WT] = 21.2128$

Some Assumptions About Happy Meal Toys

1) They Are Released At Different Times, So The Probability Of Getting Any Single Toy Is Not Equally Likely

2) You Don't Necessarily Need To Purchase A Happy Meal To Get A Happy Meal Toy!

Each Happy Meal Costs $4.99.

To Have A Complete Set, We Expect To Purchase 22 Happy Meals.

This Will Cost $109.78, Before Tax!!!

Note That Equations 1 And 2 Can Only Be Used In Certain Cases:

1) Coupons Are Collected One At A Time
2) Probability Of Collecting Any Type Coupon Is Equally Likely
3) One Complete Set Is Being Collected

What About Other Situations?

- What If We Want To Collect Two Complete Sets?
- What If The Probability Of Getting Any Single Coupon Is Not Equally Likely, i.e. A Rare Coupon?
- What If We Only Want To Collect A Subset?

- Another Method To Calculate The Waiting Time Is To Use The Dirichlet Type-II C-Integral
- This Is Used To Calculate The Lower Tail Of The Multinomial Distribution
- In This Application, The Probability That The Last Coupon Reaches Its Quota

The C-Integral Is Given By:

$$C_a^{(b)}\left(r,m\right)=\frac{\Gamma\left(m+br\right)}{\Gamma^b\left(r\right)\Gamma\left(m\right)}\int_0^{a_1}\dots\int_0^{a_b}\frac{\prod_{i=1}^{b}x_i^{r-1}dx_i}{\left(1+x_1+\dots+x_b\right)^{m+br}} \qquad (3)$$

The Expected Waiting Time Can Be Found Using The γth Factorial Moment Given By:

$$u^{[\gamma]} = \frac{b\Gamma(r+\gamma)}{\Gamma(r)\,p^{\gamma}} C_a^{(b-1)}(r, r+\gamma) \tag{4}$$

Where γ Is The Moment, b Is The Number Of Cells, And r Is The Common Quota

Hence For The Six-Sided Dice Problem, The First Moment Becomes:

$$u^{[1]} = \frac{6\Gamma(2)}{\Gamma(1)\left(\frac{1}{6}\right)} C_1^{(5)}(1,2)$$

Where The C-Integral Is Given By:

$$C_1^{(5)}(1,2) = \frac{\Gamma(7)}{\Gamma^5(1)\Gamma(2)} \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{dx_1 dx_2 dx_3 dx_4 dx_5}{(1+x_1+x_2+x_3+x_4+x_5)^7}$$

The Solution Comes Out To:

$$36 \cdot 6! \cdot \frac{49}{86400} = 14.7$$

This Is Exactly The Same Result!!!

What About A 20-Sided Die?

Using Equation 1

E[WT] = 71.9547

Using Equation 2

E[WT] = 71.4589

Using Equation 4

E[WT] = 71.9547

Consider The Experiment Of Flipping A Fair Coin Until We Have Seen Both Faces At Least Twice

Using Equation 4, The Setup Becomes:

$$u^{[1]} = \frac{2 \cdot \Gamma(3)}{\Gamma(2) \cdot \frac{1}{2}} \cdot C_a^{(1)}\left(2,3\right)$$, Where

$$C_a^{(1)}\left(2,3\right) = \frac{\Gamma(5)}{\Gamma^1(2) \cdot \Gamma(3)} \cdot \int_0^1 \frac{x_1}{(1+x_1)^5} dx_1$$

The Answer Is:

$$8 \cdot \frac{4!}{2} \cdot \frac{11}{192} = 5.5$$

Note That This Is Not The Same As Simply Doubling The Waiting Time For Collecting One Complete Set

Consider The Experiment Of Rolling A Fair Die Until We Have Seen All Six Faces At Least Twice

Using Equation 4, The Setup Becomes:

$$u^{[1]} = \frac{6\cdot\Gamma(3)}{\Gamma(2)\cdot\frac{1}{6}}\cdot C_a^{(5)}(2,3)$$, Where

$$C_a^{(5)}(2,3) = \frac{\Gamma(13)}{\Gamma^5(2)\cdot\Gamma(3)}\cdot\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1x_2x_3x_4x_5}{(1+x_1+x_2+x_3+x_4+x_5)^{13}}dx_1dx_2dx_3dx_4dx_5$$

The Answer Is 24.1338

(Note That Is Not The Same As Doubling The Expected Waiting Time For Collecting One Complete Set)

What If You Had Two Children, And Each Child Wanted A Complete Set Of The 8 Happy Meal Toys?

Using Equation 4, You Would Need To Buy 34.8846 Happy Meals!

Now, Let’s Consider Coupons Whose Probability Are Not Uniform – For Example, A Rare Coupon.

Consider The 8 Happy Meal Toys. If Each Toy Is Equally Likely To Be Obtained, The Probability is 1/8, Or 12.5% For Each Toy.

Let's Say One Toy Has A Probability Of 1/100 Of Being Obtained. Then The Remaining 7 Toys Have A 99/700, Or 14.14% Chance Of Being Obtained.

This Waiting Time Is 102.1908!!!

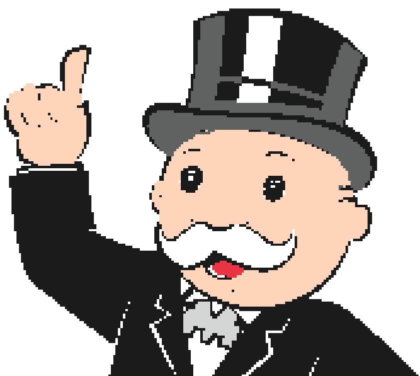
The McDonald's Monopoly Game Requires Collection Of A Subset To Win Certain Prizes, i.e. We Do Not Have To Collect ALL The Coupons Available

Each Subset Consists Of One Rare Coupon While The Other Coupons In The Subset Are Easily Obtainable

MONOPOLY

COLLECT & WIN!

Complete Winning Combinations to win awesome prizes!



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Stamps: 512, 513 & 514

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Collect All 3 Stamps
Stamps: 515, 516 & 517

Cessna Private Jet Trip
(2 available)[3]
Collect All 3 Stamps
Stamps: 518, 519 & 520

$2,500
and a smartphone with Mobile Wallet (238 available)[4]
Collect All 3 Stamps
Stamps: 509, 510 & 511

$10,000
Cash Prize to Help Get Your Bills Paid
(4 available)[5]
Collect Both Stamps
Stamps: 527 & 528

$5,000
Target Shopping Experience with Early Access on Black Friday (10 available)[6]
Collect All 4 Stamps
Stamps: 523, 524, 525 & 526

$5,000
Cash Prize (5 available)[7]
Collect All 3 Stamps
Stamps: 506, 507 & 508

Fuel for a Year
(4 available)[8]
Collect All 3 Stamps
Stamps: 503, 504 & 505

$50
Cash Prize (1,000 available)[9]
Collect Both Stamps
Stamps: 501 & 502

$1,000,000
Payable $50,000/yr for 20 yrs, no interest
(1 available)[10] Collect Both Stamps
Stamps: 521 & 522

INCOME TAX

LUXURY TAX

CHANCE

JUST VISITING

COLLECT $200 SALARY AS YOU PASS GO

What’s Next??

Calculating The Waiting Time When Collecting A Subset (Equally Likely Scenario And Otherwise)

Questions?

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Special Thanks to:
Dr. Hokwon Cho, Associate Professor, UNLV

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