Euler's Multiple Solutions to a Diophantine Problem

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18 April 2015

Leonhard Euler (1707-1783)

- Swiss
- Had 13 kids
- Worked in St. Petersburg and Berlin
- By 1735, blind in right eye – went totally blind later, but kept writing (secretary)
- Published 530 books and papers in his life, and many more after his death (including the ones we will consider)
- Very prolific and successful, but also not always rigorous

Graphic from http://sebastianiaguirre.wordpress.com/2011/04/12/project-euler/

Some of Euler's Mathematics

1. Using certain notations: $f(x)$, e, $\sum$, i
2. Using a, b, c for the sides of a right triangle
3. $e^{ix} = \cos x + i \sin x \quad [e^{i\pi} + 1 = 0]$
4. $V - E + F = 2$, Ex: cube (8 vertices, 12 edges, 6 faces)

More of Euler's Mathematics

1. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$
2. Euler line (geometry)
3. Euler's method (ordinary differential equations)
4. Eulerian path (graph theory)

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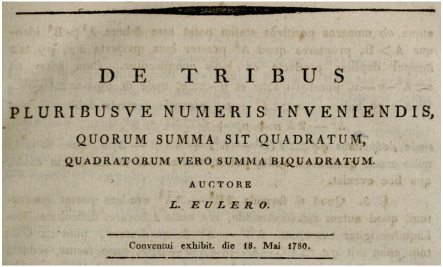
- "On finding three or more numbers, the sum of which is a square and the sum of the squares of which is a fourth power" (1824).
- Objective: understand Euler's solution and follow his algebraic twists and turns along the way.



DE TRIBUS

PLURIBUSVE NUMERIS INVENIENDIS,

QUORUM SUMMA SIT QUADRATUM,

QUADRATORUM VERO SUMMA BIQUADRATUM.

AUCTORE

L. EULERO.

Conventui exhibit. die 18. Mai 1780.

Graphic from http://eulerarchive.maa.org/

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- Which is bigger, M or N? Why?

My translations are NOT literal, but get the point across.

Patterns and facts

- $(s+t)^2 = s^2 + 2st + t^2$
- $(s+t+u)^2 = s^2 + t^2 + u^2 + 2st + 2su + 2tu$
- In the quadratic $ax^2 + bx + c = 0$, the sum of the two roots is $-\frac{b}{a}$.

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- Euclid showed that EVERY primitive Pythagorean triple can be put into this form, for some choice of p and q.

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- In addition, $a^2 + b^2$ should be a square, which happens in the same way by setting $a = p^2 - q^2$ and $b = 2pq$: from here, it follows that $x^2 + y^2 = (a^2 + b^2)^2 = (p^2 + q^2)^4$, and thus the latter condition has now been fully satisfied. [**]

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- Then, it remains to satisfy the first condition, namely that $x + y$ be a square."

Euler's solution: §6

- "From these facts it is found that

$$x = a^2 - b^2 = p^4 - 6p^2q^2 + q^4 \quad \text{and} \quad y = 2ab = 4p^3q - 4pq^3;$$

and so the following formula $[x + y]$ ought to be a square

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[with $p > q > 0$ and $a > b$]."

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- Why do we have to pick $p > q$? Why do we have to have $a > b$?

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- "The formula is solved by setting $\sqrt{x+y} = p^2 - 2pq + q^2$, from which $\frac{p}{q} = \frac{3}{2}$, or $p = 3$ and $q = 2$."

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$$(p^2 - 2pq + q^2)^2 = p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4,$$

which doesn't quite equal $p^4 + 4p^3q - 6p^2q^2 - 4pq^3 + q^4$, as he claimed.

Euler's solution: §7 (cont.)

But he is close. Three of the terms are identical, and the other two just have different signs. So, let's set the two expressions equal and see what happens.

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- But then $a = 5$, and $b = 12$, and so $x < 0$, and this solution is rejected." [$x = -119; y = 120$]

More of Euler's Algebra Skills

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$$\begin{aligned} p^4 &= 81 + 108v + 54v^2 + 12v^3 + v^4, \\ 4p^3q &= 216 + 216v + 72v^2 + 8v^3, \\ 6p^2q^2 &= 216 + 144v + 24v^2, \\ 4pq^3 &= 96 + 32v, \\ q^4 &= 16. \end{aligned}$$

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Euler then guesses the square root of this to be: $1 + 74v - v^2$. Why?

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$$102v^2 + 20v^3 \quad = \quad 5474v^2 - 148v^3$$

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$$p = 1469 \quad \text{and} \quad q = 84.$$

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"...which are the same that Fermat, and others after him, found. The sum of them is the square of the number 2,372,159, while the sum of the squares is the fourth power of the number 2,165,017."

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WOW!!! But wait, there's more ....

Finding a Pattern, I

Next, Euler finds three numbers (x, y, z).

1. Set $x = a^2 + b^2 - c^2$, $y = 2ac$, $z = 2bc$.
2. Then set $a = p^2 + q^2 - r^2$, $b = 2pr$, $c = 2qr$.
3. Then GUESS the square root of $x + y + z$.
4. ...Euler finds $p = r + \frac{3}{2}q$.
5. He then chooses $q = 2$, $r = 1$ to get $p = 4$ and thus ...
6. "$x = 409$; $y = 152$; $z = 64$, the sum of which is $x + y + z = 625 = 25^2$; while the sum of the squares will be $xx + yy + zz = 194{,}481 = 441^2 = 21^4$." (MUCH SMALLER)

Finding a Pattern, II

Next, Euler finds four numbers (x, y, z, v).

1. Set $x = a^2 + b^2 + c^2 - d^2$, $y = 2ad$, $z = 2bd$, $v = 2cd$.
2. Then set $a = p^2 + q^2 + r^2 - s^2$, $b = 2ps$, $c = 2qs$, $d = 2rs$.
3. Then GUESS the square root of $x + y + z + v$.
4. ... Euler finds $p = s + \frac{3}{2}r - q$.
5. He then chooses $r = 2$, $q = s = 1$ to get $p = 3$ and thus ...
6. "$x = 193$; $y = 104$; $z = 48$; $v = 16$, the sum of which is $x + y + z + v = 361 = 19^2$; while the sum of the squares will be $xx + yy + zz + vv = (pp + qq + rr + ss)^4 = 15^4$."

Finding a Pattern, III

Next, Euler finds five numbers (x, y, z, v, w).

1. Set $x = a^2 + b^2 + c^2 + d^2 - e^2$, $y = 2ae$, $z = 2be$, $v = 2ce$, $w = 2de$.
2. Then set $a = p^2 + q^2 + r^2 + s^2 - t^2$, $b = 2pt$, $c = 2qt$, $d = 2rt$, $e = 2st$.
3. Then GUESS the square root of $x + y + z + v + w$.
4. ... Euler finds $p = t + \frac{3}{2}s - r - q$.
5. He then chooses $s = 2$, $t = r = q = 1$ to get $p = 2$ and thus ...
6. "$x = 89$; $y = 72$; $z = 32$; $v = 16$; $w = 16$, the sum of which is $x + y + z + v + w = 225 = 15^2$; while the sum of the squares will be $x^2 + y^2 + z^2 + v^2 + w^2 = 11^4$."

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  You try it!!
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  You try it!!
- For 6 numbers, $p = u + \frac{3}{2}t - s - r - q$.

Student Work

- I teach Topics in the History of Mathematics. I assign a project in which students have to engage with a primary source or a translation of a primary source.
- One student chose this paper.
- She found six numbers that had the same property. Namely: 97, 112, 64, 64, 64, and 128.
- Their sum is $529 = 23^2$, and the sum of their squares is $50625 = 15^4$.

Another solution

"On a notable advancement in Diophantine analysis" (1830) has a different solution. Why?

Another solution

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...because Lagrange criticized Euler's original solution method. So Euler wrote two more papers going into more generality about how to generate solutions.

E772

Euler generalizes to find integer solutions to

$$a^2x^4 + 2abx^3y + cx^2y^2 + 2bdxy^3 + d^2y^4 = \square$$

by making substitutions and taking advantage of certain patterns. We'll work through an example.

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But first, we'll set $y = 1$ and look for rational solutions. Why is this OK?

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then

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So now we have to find solutions to:

$$1 + 6pq + p^2q^2 = p^2 + 8q^2,$$

which is quadratic in p or in q.

Quadratics

As a quadratic in p, the equation is

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As a quadratic in q, the sum of the roots is

$$q + q' = \frac{-6p}{p^2 - 8}.$$

Note that if $q = 0$, then $p = 1$. Also, if $q = 1$, then $p = \frac{7}{6}$.

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Also, if $q = 1$, then $p = \frac{7}{6}$. Then $q' = \frac{-7}{\frac{49}{36} - 8} - 1 = \frac{13}{239} \ldots.$

So $x = 0; \frac{6}{7}; \frac{1434}{91}; \ldots;$ or $x = \frac{7}{6}; \frac{91}{1434}; \ldots.$

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"...and in this way hardly any doubt can survive, but that all the satisfactory values for x will clearly be rooted out."

But is this **REALLY** all the solutions?

Thank you!

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