**Neither Div nor Curl nor Both Constitute the Derivative**

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**Abstract**

The Div and Curl operators are derivative-like but are only aspects of the derivative of a map from 3-space to 3-space. The derivative of such a given map is a linear map from 3-space to 3-space. By employing the Frechet difference quotient at the outset, we can construct that derivative without use of the heavy machinery of advanced analysis. The process, results, and properties parallel those of the one-dimensional case.

**Attribution**

1. H. K. Nickerson, D. C. Spencer, and N. E. Steenrod, *Advanced* *Calculus*, Dover Publications, 2011. Unabridged republication of the work originally published in 1959 by the D. Van Nostrand Company.
2. Spivak, Michael, *Calculus on Manifolds*: *A Modern Approach to Classical Theorems of Advanced Calculus*,” W. A. Benjamin, Inc., New York, Amsterdam, 1965.
3. H. M. Schey, *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus*, W.W. Norton & Company, Inc., New York, 1973.

**Outline**

* Alternate definition of the derivative of a real-valued function of a real variable (the case)
* The Frechet derivative of a scalar-valued function of a vector (the case)
* The Frechet derivative of a vector-valued function of a vector (the case)
* The divergence and curl of a vector field
* An application: the divergence and curl of a steady flow
* Generalizations and the chain rule

**Scalar-valued Functions of a Scalar (1)**

Situation: represents the real numbers. .

Goal: Define the derivative of at .

For consider the following limit:

If this limit exists, then is called “the derivative of at with respect to ”

**Scalar-valued Functions of a Scalar (2)**

Claim: is linear in the variable

Pick and fix in and consider . Show first that = . Well,

.

Therefore,

**Scalar-valued Functions of a Scalar (3)**

Now let and denote two non-zero elements of and consider .

for some

 .

So, is linear in the second variable.

**Scalar-valued Functions of a Scalar (4)**

**Definition**:

The derivative of at , denoted by , is the linear map from defined by

**Examples (old chestnuts in new shells):**

1. is a constant function, , where is a constant. Then , i.e. , sends all to zero for every

Argument: .

1. is a linear map, i.e., , where is an arbitrary but fixed number.

Then is a constant map, i.e.,

 for every .

 Argument: .

**Scalar-valued Functions of a Scalar (5)**

**Examples (old chestnuts in new shells), cont’d:**

3. implies that , i.e.,

 all

Argument:

Now, divide by and pass to the limit to see that

 .

**Scalar-valued Functions of a Scalar (6)**

**Best affine approximation to**

Pick and fix . Then the mean value theorem says that

.

You will see direct parallels to this equation as we work our way up to functions from .

**Scalar-valued Functions of a Vector (1)**

Situation:

Goal: Define the derivative of at , . For , consider the following limit that defines function .

If this limit exists, then is called “the derivative of at with respect to ”.

**Scalar-valued Functions of a Vector (2)**

Claim: is linear in the variable .

First, show that for .

Pick and fix in . If , the difference quotient is . If , then

**Scalar-valued Functions of a Vector (3)**

Now, consider . We must show that ,

where

.

There exists a version of the mean value theorem that we will need:

If exists for , then there exists such that

This version can be established fairly easily from the standard version.

In the proof of linearity for the case I used the fact that for any two non-zero real numbers there exists a third, , say, such that . The above version of the mean value theorem is used similarly to complete the proof of linearity for the present case. The details, somewhat involved, appear on page 5 of the handout.

So, is linear in for every .

**Scalar-valued Functions of a Vector (4)**

**Definition:**

The derivative of at is the linear map

defined by

**Scalar-valued Functions of a Vector (5)**

The Gradient

 Let

 Expand in the standard basis.

Invoke the fact that is linear in and expand via ’s components

Arrive at

A theorem from linear algebra says that if is linear, then there exists a unique vector such that

So , which implies that

Hence,

**Scalar-valued Functions of a Vector (6)**

Observe that knowing is equivalent to knowing , but and

**Best affine approximation to**

Pick and fix Then

.

 Or, in terms of the gradient,

**Vector-valued Functions of a Vector (1)**

Situation:

For , consider the following limit that defines :

if the limit exists.

Examples:

1. is a constant function.

 for every . is a fixed vector.

 Then .

 So, .

**Vector-valued Functions of a Vector (2)**

2. is a linear map.

 , where is linear.

 Then .

 .

Now, divide by and pass to the limit to see that

 for all

**Vector-valued Functions of a Vector (3)**

Claim: is linear in the variable .

Let represent the usual basis in . Then

 ,

where the represent the component functions. Let represent the derivative with respect to of the component functions. Earlier I sketched an argument that showed that the are linear in the variable . It can be shown that inherits linearity in from the . So,

is linear in the variable .

**Definition**:

The derivative of at is the linear transformation

defined by

**Vector-valued Functions of a Vector (4)**

Examples (restatement in terms of rather than ):

1. The derivative of a constant function is the zero vector. So transforms all of into the zero vector.
2. The derivative of a linear function is a constant.

 linear, implies that

 for every .

Now we can get back to more familiar, classical ground. is a linear map. Given a basis, a linear map can be represented by a matrix relative to that basis. So, as above, let represent the standard basis. Then

 ,

where the scalar-valued are the component functions of .

Then

**Vector-valued Functions of a Vector (5)**

I have here invoked both the fact (shown earlier) that, for functions and the linearity of to arrive at Eqn..

Observe from Eqn. that, relative to the standard basis, is represented by

where here and below I use to indicate the matrix of .

**Best affine approximation to *F***

As in the previous and cases, we have a best approximation via Taylor’s formula:

**Divergence and Curl of a Vector Field (1)**

Given , then is a linear map from . Linear maps may be broken into the sum of their symmetric and skew-symmetric parts.

Let represent the adjoint of . Then

 and

 *skew-symmetric part*.

Observe that sum of the two parts.

**Divergence and Curl of a Vector Field (2)**

To simplify typography, until further notice let

Then the matrix representation of is

and the matrix of is

**Divergence and Curl of a Vector Field (3)**

Then a bit of matrix algebra leads to the matrix representations of and

They are

**Divergence and Curl of a Vector Field (4)**

**Definition:**

The Divergence of is a map , defined by

The *Trace* of is equal to the sum of the roots of the characteristic polynomial of , a property of the linear map that is invariant to matrix representation. The trace of any linear map from is equal to the sum of the diagonal elements of any matrix representation. Hence, the familiar version of the Divergence:

Observe also that

**Divergence and Curl of a Vector Field (5)**

**Definition:**

The Curl of at is defined by

How do we know that a vector that makes the right side of Eqn. equal to the left-hand side exists? Well, is a skew-symmetric transformation and item A5 of the Appendix of the handout asserts that such a vector exists for such maps; we name it . Also, according to A5, if the matrix of a skew-symmetric map looks like

then the vector at issue ( here) is

**Divergence and Curl of a Vector Field (6)**

Compare the matrix representation of and you see that

or, in the usual notation,

**Divergence and Curl of a Steady Flow: an application (1)**

Consider a vector field and the flow induced by the differential equation

Note that time does not appear on the right hand side of Eqn. , hence “steady flow.”

Pick and fix . Expand around via Taylor’s formula; substitute the result into Eqn. to yield

Split into the sum of its symmetric and skew-symmetric components

and substitute into Eqn. to obtain

**Divergence and Curl of a Steady Flow: an application (2)**

Now, consider the motion of the points within a small sphere centered at at time zero, say. How has that spherical test ball changed in a small time ? The terms on the right-hand side yield an approximate answer.

Consider first the motion induced by

Equation says that the test ball rotates about an axis through parallel to with angular velocity . That motion is volume-preserving.

**Divergence and Curl of a Steady Flow: an application (3)**

Consider next the motion described by

 is a symmetric map centered at . Let represent its eigenvalues and denote the corresponding orthonormal set of eigenvectors. Then the corresponding eigenvalues of the flow described by Eqn. are . That flow distorts the spherical ball slightly into an ellipsoid. The volume of the ball changes by a factor that is the determinant of the transformation. That determinant is and its time rate of change when is .

Finally, consider the motion induced by

Equation says that the ellipsoid experiences a small translation with velocity vector .

**The General Case and the Chain Rule (1)**

The chain rule is so important in differential calculus that its multidimensional version deserves a brief discussion. Before doing so, I point out that our restriction to the case above was not necessary; I invoked the restriction because of the emphasis on the undergraduate calculus sequence.

Let and represent natural numbers and consider . If is differentiable at , then its derivative can be defined via the Frechet difference quotient just as we did in the earlier cases. The process is identical to the one used for the case. The derivative is a linear transformation as before, and all goes through except the discussion of the Curl, which is unique to the three-dimensional case. With this generalization all cases are subsumed, again except for the Curl, by this general case.

**The General Case and the Chain Rule (2)**

Consider now the chain rule in the general case. Let and represent natural numbers. Situation: is differentiable at , and is differentiable at , then is differentiable at and

where represents function composition. Observe that in Eqn. is a linear map and is a linear map; hence the composition of the two maps is a linear map, and all is well. Use brackets to denote matrix representation, and the matrix version of Eqn. becomes

**The General Case and the Chain Rule (3)**

Recall the one-variable chain rule.

where the juxtaposition on the right-hand side of Eqn denotes multiplication, but could just as well have been written

because, in the one-dimensional case, composition of linear functions is multiplication. Compare Eqn. and Eqn. and you see that Eqn. is just a special case of Eqn. .

**Nerd’s Bumper Sticker**

And God said

and there was light.

**Closure**

* **The derivative is a linear map.**
* **The Divergence and Curl are aspects of that map that account for about 4/9 of its properties, an especially important fraction in applications.**

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**“Some people can stay longer in an hour than others can in a week.”**

**William Dean Howells, American novelist, 1837-1920**