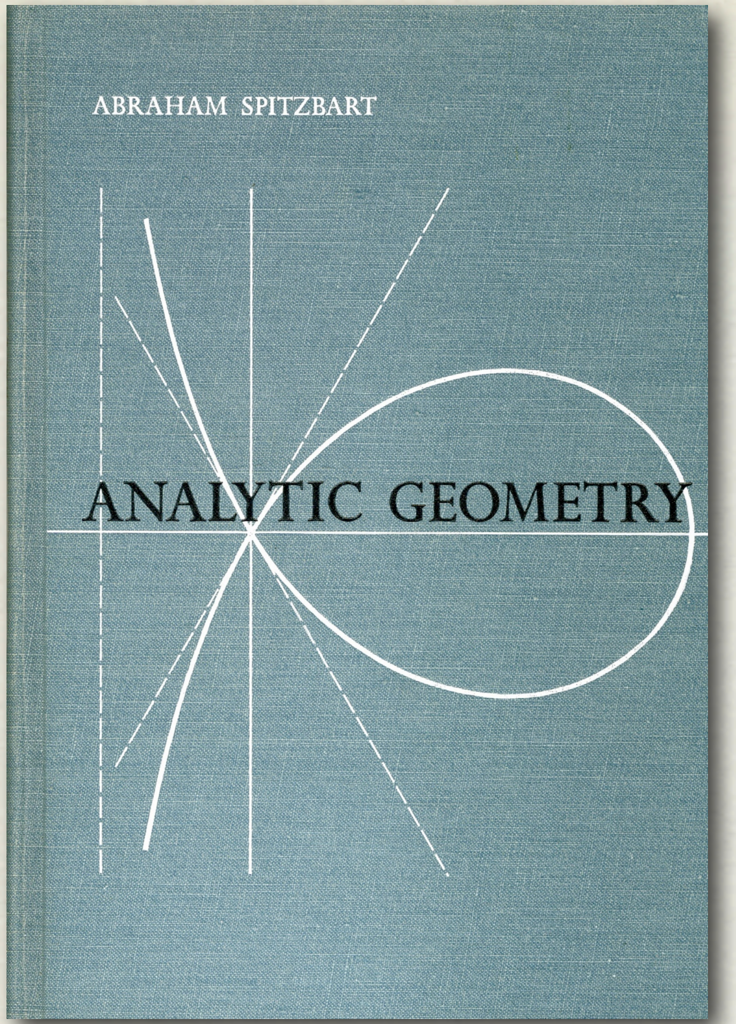
SOME IRRATIONALS I HAVE KNOWN

John Martin

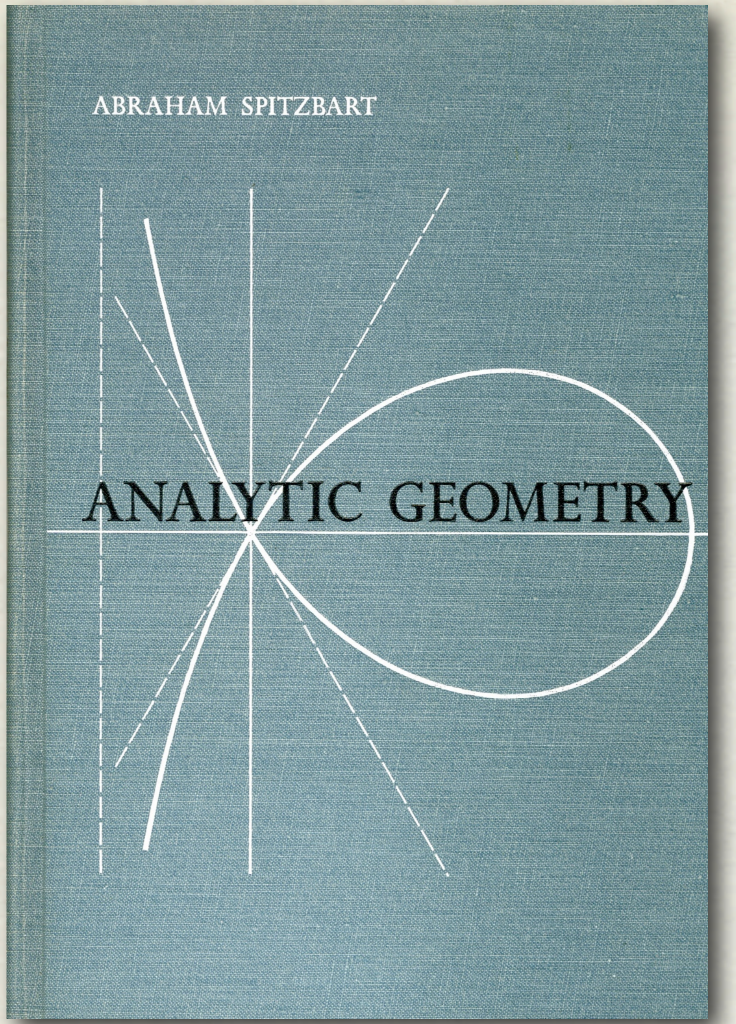
Santa Rosa Junior College


HUMBOLDT STATE UNIVERSITY



Dr. Orval Klose

$\sqrt{2}$ **and** π



Dr. Orval Klose

An Extraordinary Statement

"It may surprise you to learn that the set of irrationals is more numerous than the set of rationals."

The Number System

Ishango Bone
c. 20,000 B.C.

Lemombo Bone
c. 35,000 B.C.

The Number System

$$\{1,\ 2,\ 3,\ \ldots\}$$

The **Natural** Numbers $(\mathbb{N})$

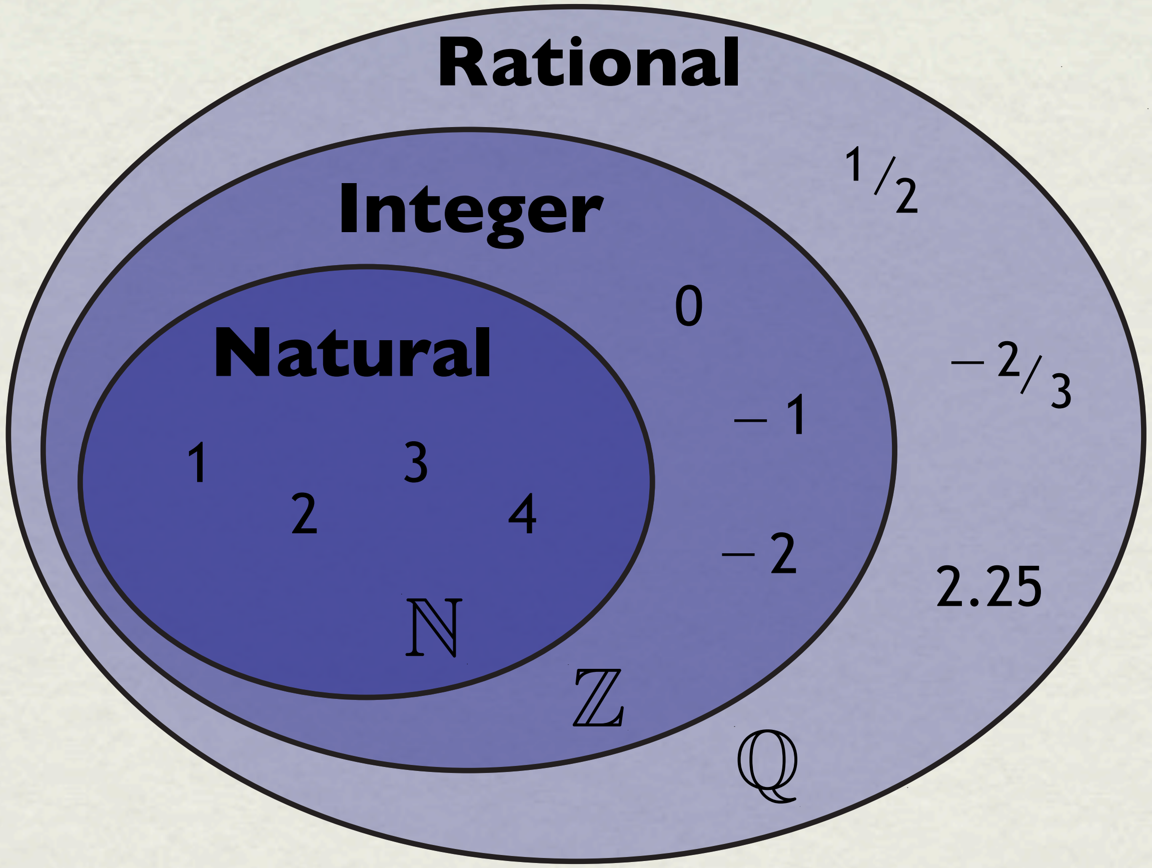
$$\{\ldots\ ,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}$$

The **Integers** $(\mathbb{Z})$

$$\left\{ \frac{a}{b} \mid a \text{ and } b \text{ are integers and } b \neq 0 \right\}$$

The **Rational** Numbers $(\mathbb{Q})$

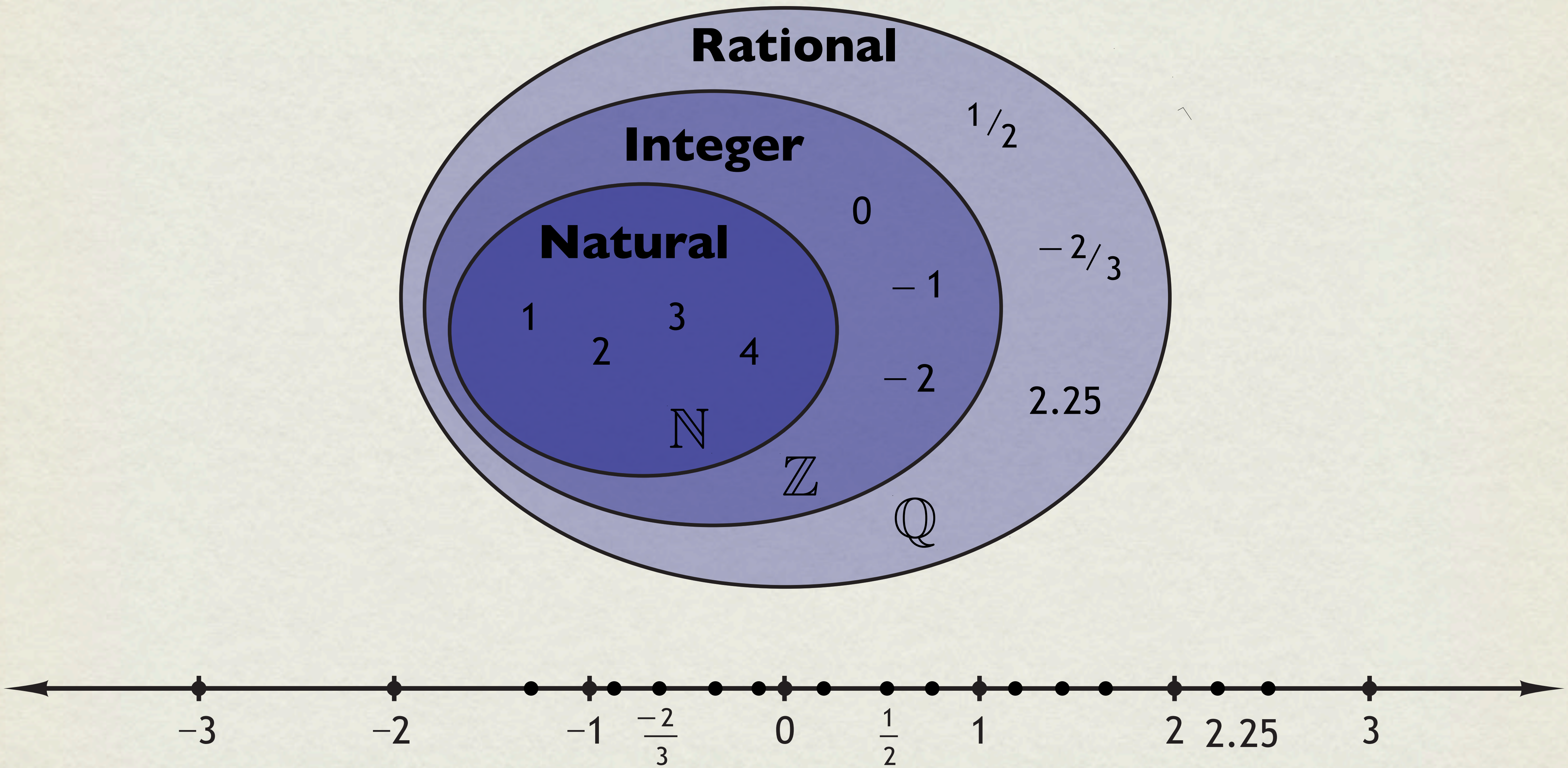
The Number System



The naturals are a **subset** of the integers. $\mathbb{N} \subset \mathbb{Z}$

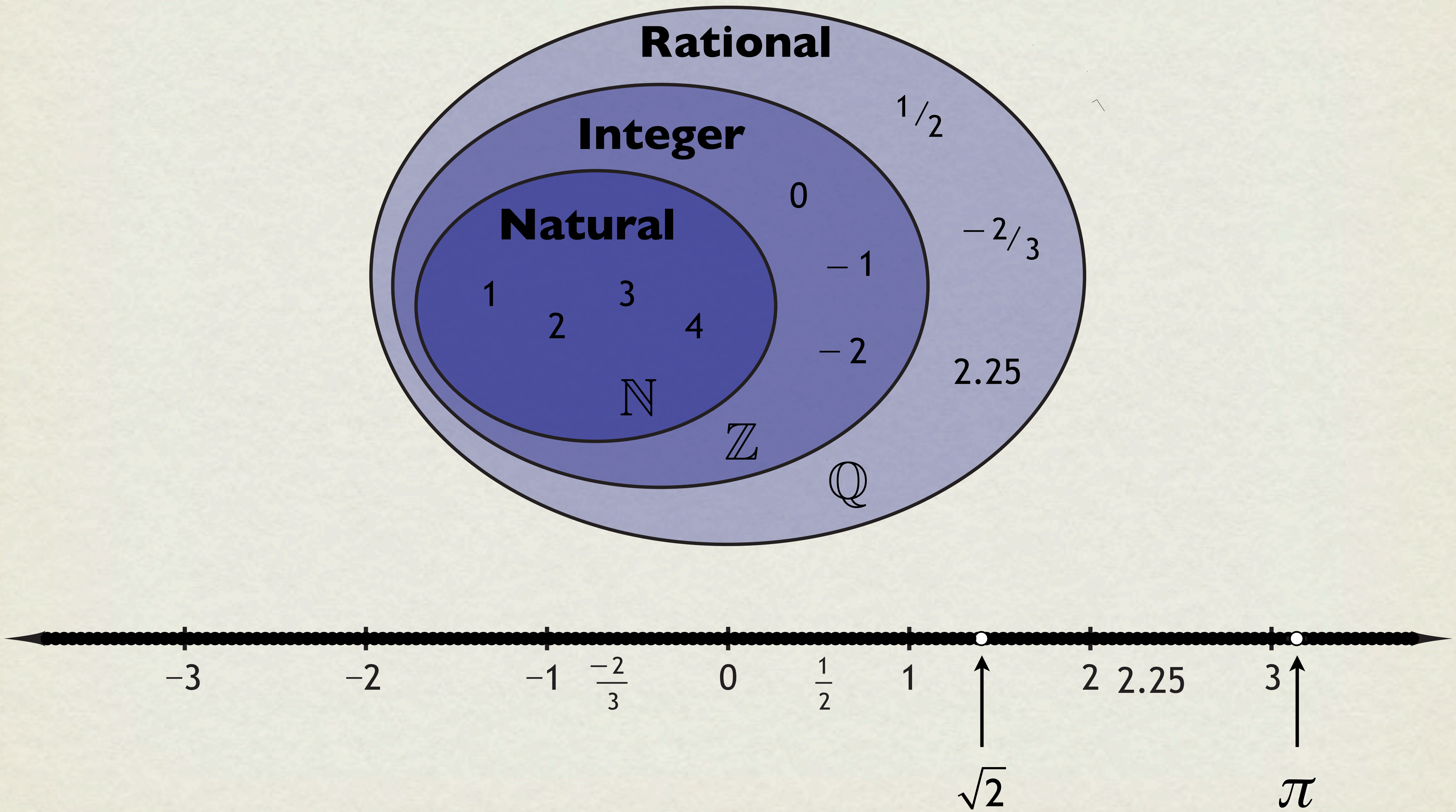
The integers are a subset of the rationals. $\mathbb{Z} \subset \mathbb{Q}$

The Number System

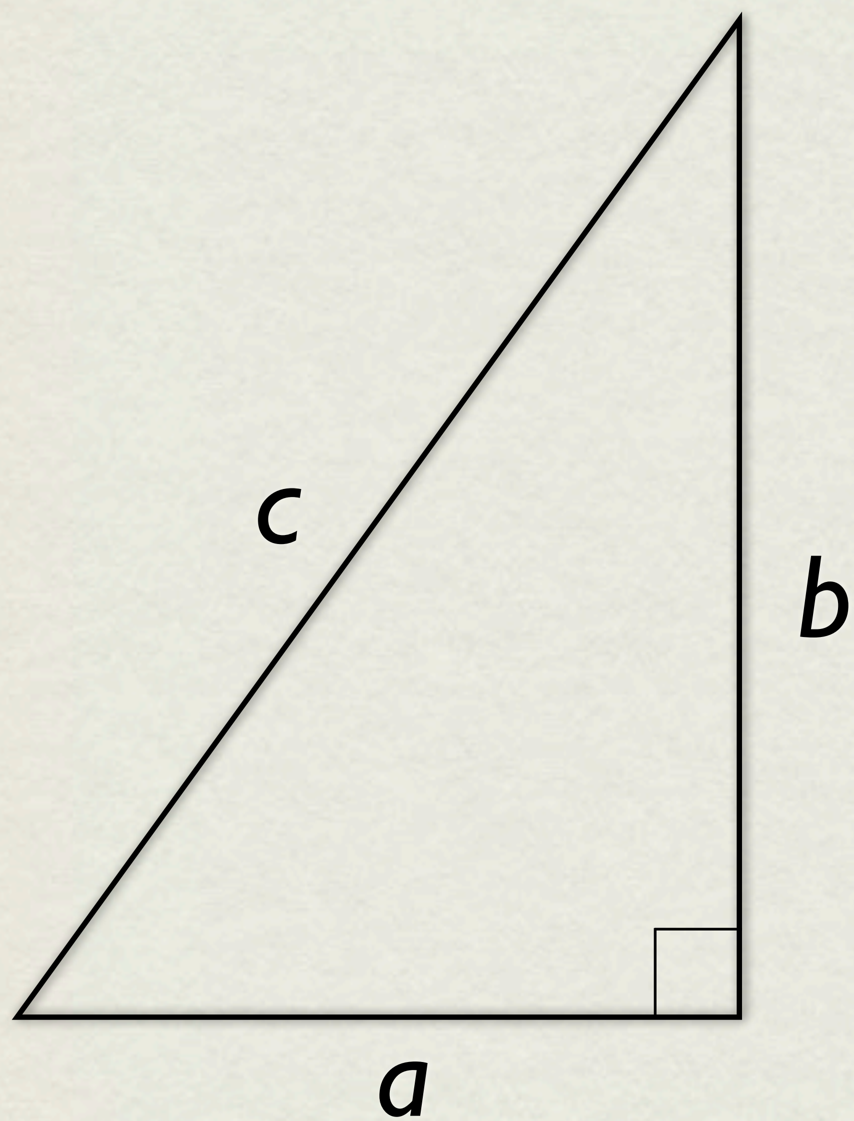


Dense: Between any two fractions lies another.

The Number System



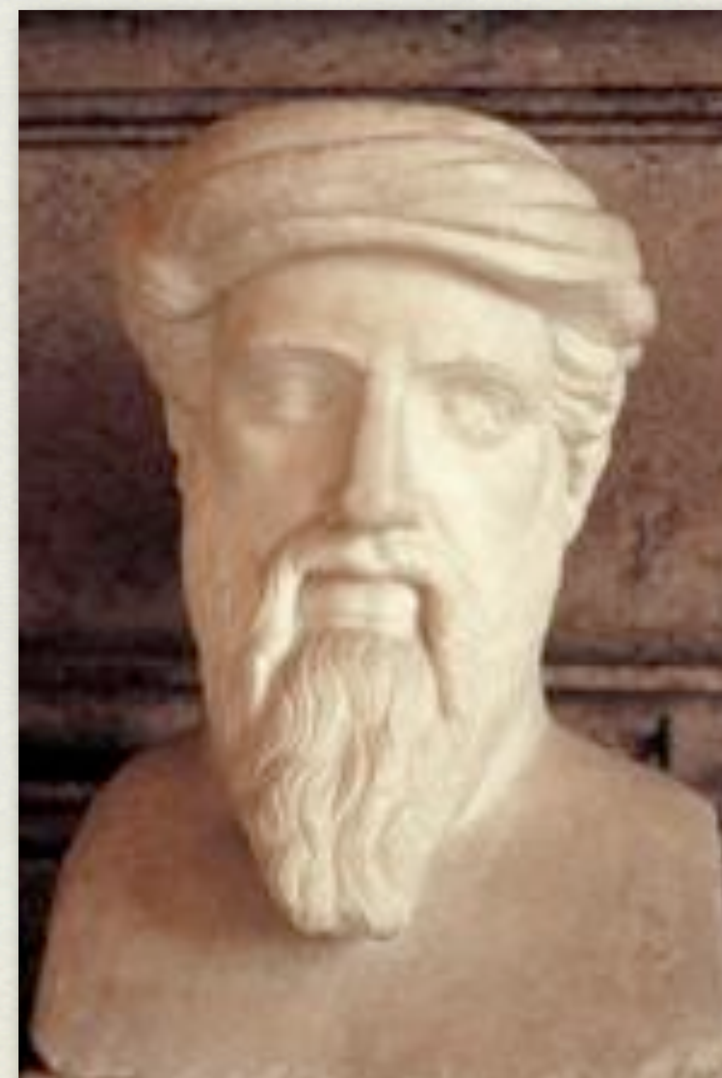
A Small Problem



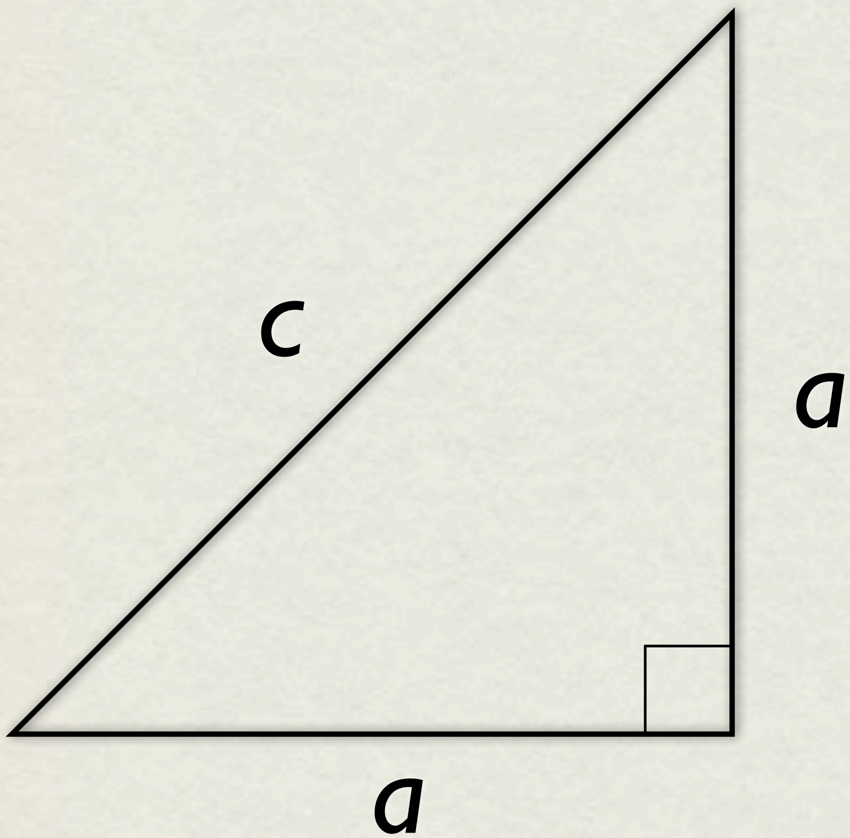
$$a^2 + b^2 = c^2$$

Pythagoras of Samos

c. 570 B.C. – c. 495 B.C.



A Small Problem



$$a^2 + a^2 = c^2$$

If $a = 1$ then

$$c^2 = 1^2 + 1^2 = 2$$

and

$$c = \sqrt{2}.$$

What kind of number is $\sqrt{2}$?

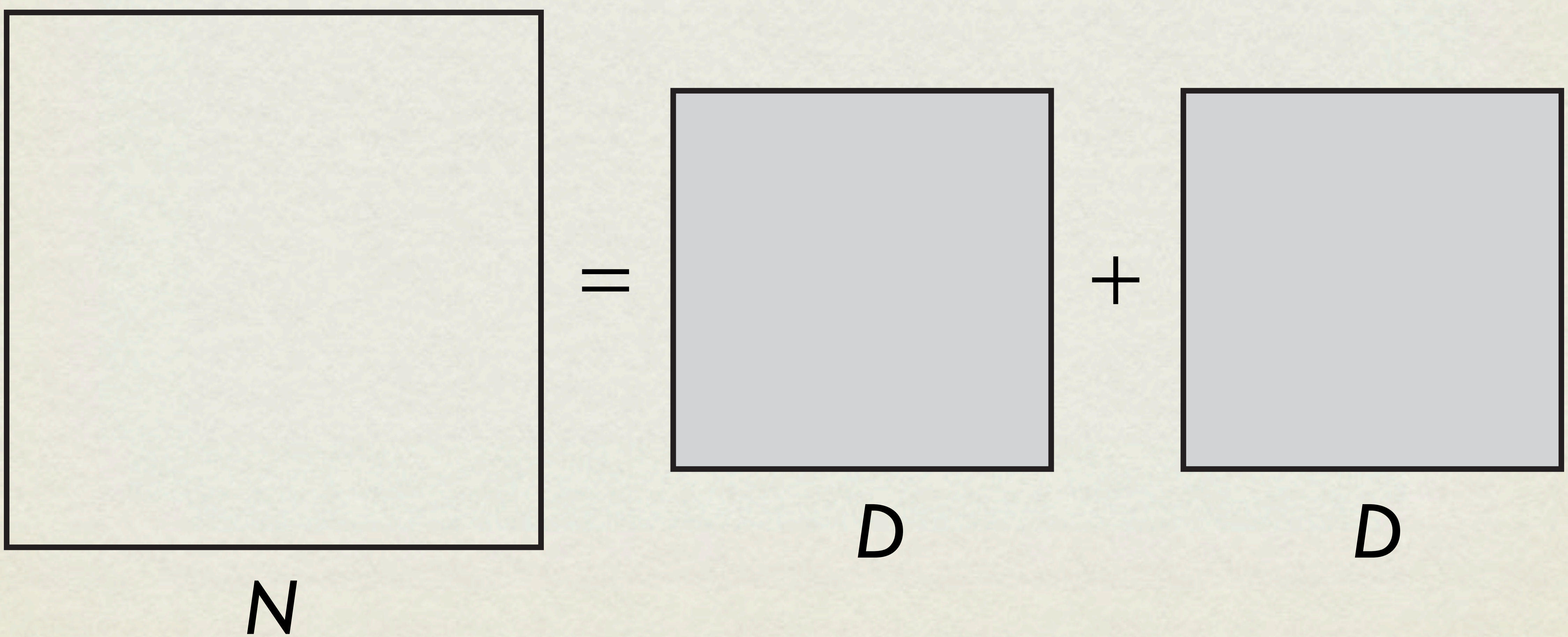
A Small Problem

$$\sqrt{2} \in \mathbb{Q}$$

Hippasus

Proof That $\sqrt{2}$ is Not Rational

Assume that $\sqrt{2} = \dfrac{N}{D}$, where N and D have no common factor. Then $2D^2 = N^2$.



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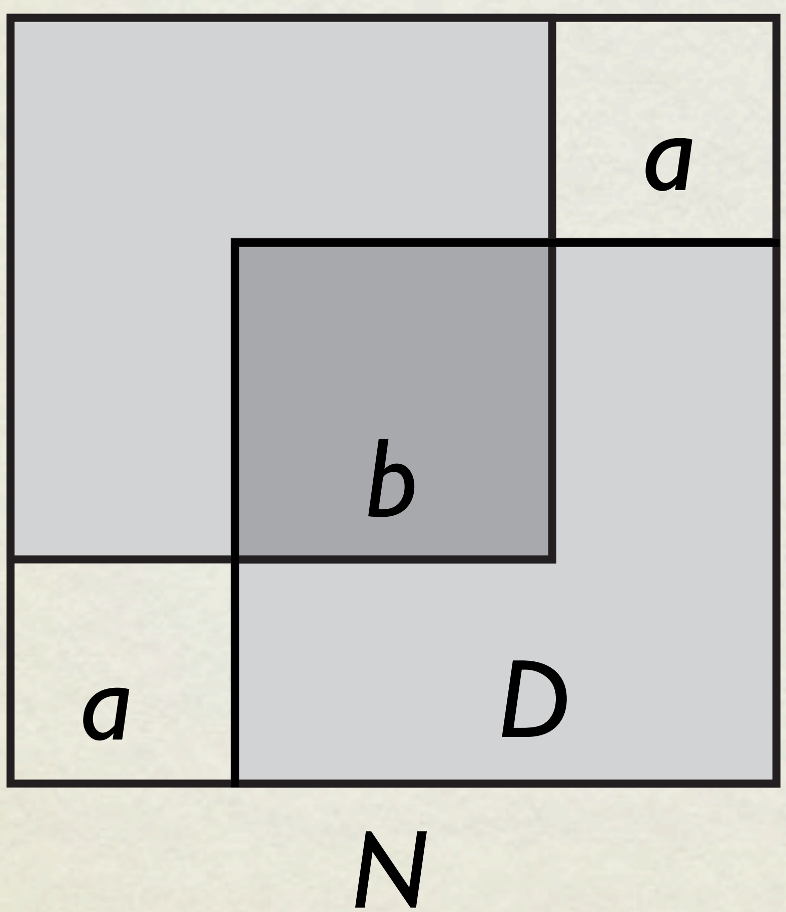
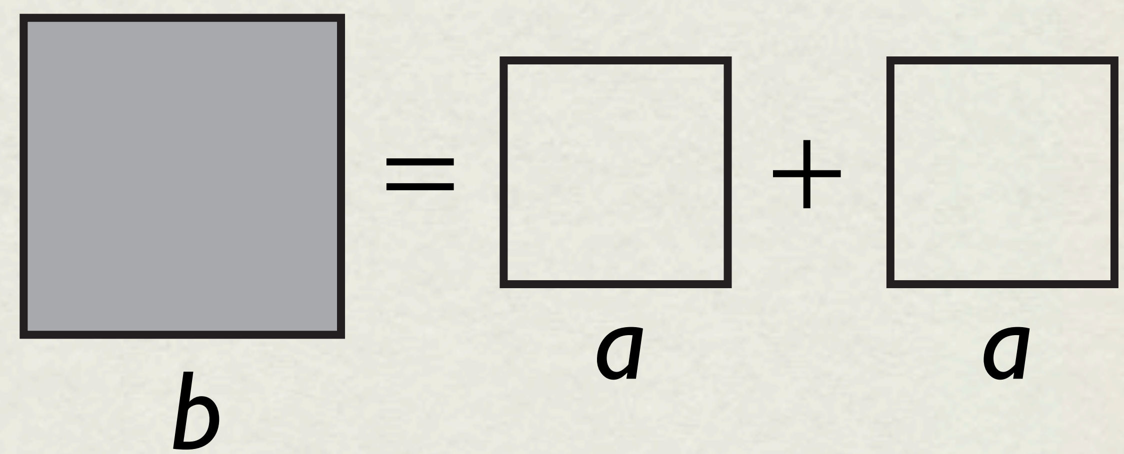




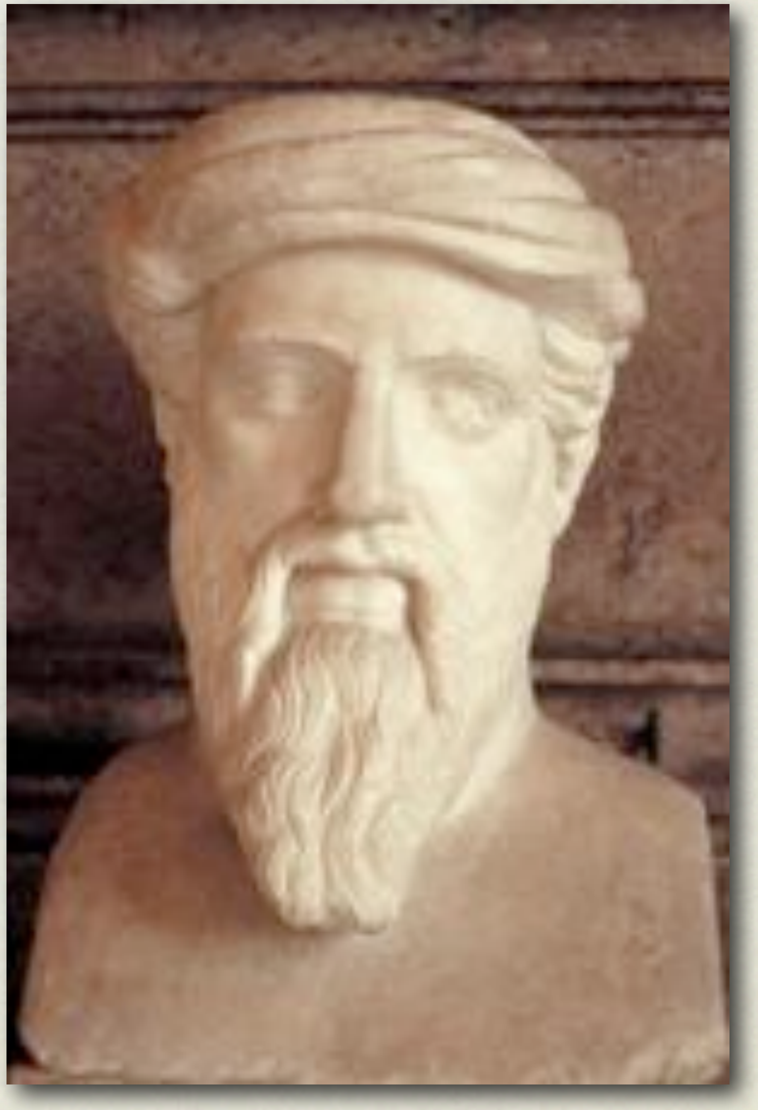
So, $b^2 = a^2 + a^2 = 2a^2$

Contradiction! $\sqrt{2} \notin \mathbb{Q}$

A Small Problem

Pythagoras of Samos

c. 570 B.C. – c. 495 B.C.

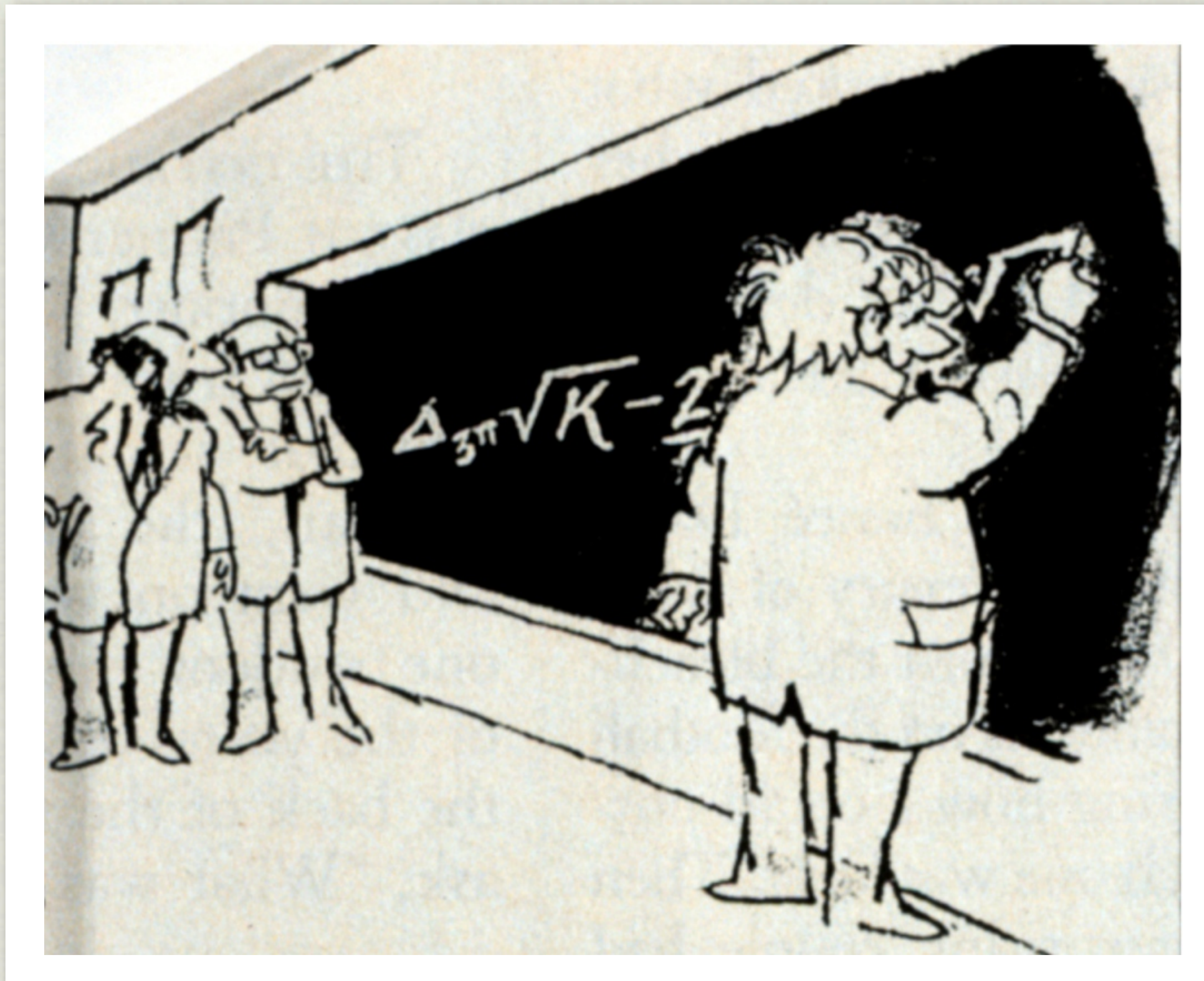


“Alogos”

The Unutterable

A real number that cannot be expressed as the ratio of two integers is said to be **irrational**.

Humor in Mathematics?



“We have reason to believe that Martin himself is an irrational number!”

The Golden Ratio

Numerical Constants

MISCELLANEOUS CONSTANTS

π CONSTANTS

π = 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510
$1/\pi$ = 0.31830 98861 83790 67153 77675 26745 02872 40689 19291 48091
π^2 = 9.86960 44010 89358 61883 44909 99876 15113 53136 99407 24079
$\log_e \pi$ = 1.14472 98858 49400 17414 34273 51353 05871 16472 94812 91531
$\log_{10} \pi$ = 0.49714 98726 94133 85435 12682 88290 89887 36516 78324 38044
$\log_{10} \sqrt{2\pi}$ = 0.39908 99341 79057 52478 25035 91507 69595 02099 34102 92127

CONSTANTS INVOLVING e

e = 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995
$1/e$ = 0.36787 94411 71442 32159 55237 70161 46086 74458 11131 03176
e^2 = 7.38905 60989 30650 22723 04274 60575 00781 31803 15570 55184
$M = \log_{10} e$ = 0.43429 44819 03251 82765 11289 18916 60508 22943 97005 80366
$1/M = \log_e 10$ = 2.30258 50929 94045 68401 79914 54684 36420 76011 01488 62877
$\log_{10} M$ = 9.63778 43113 00536 78912 29674 98565 − 10

π^e AND e^π CONSTANTS

π^e = 22.45915 77183 61045 47342 71522
e^π = 23.14069 26327 79269 00572 90864
$e^{-\pi}$ = 0.04321 39182 63772 24977 44177
$e^{\frac{1}{2}\pi}$ = 4.81047 73809 65351 65547 30357
$i^i = e^{-\frac{1}{2}\pi}$ = 0.20787 95763 50761 90854 69556

NUMERICAL CONSTANTS

$\sqrt{2}$ = 1.41421 35623 73095 04880 16887 24209 69807 85696 71875 37694
$\sqrt[3]{2}$ = 1.25992 10498 94873 16476 72106 07278 22835 05702 51464 70150
$\log_e 2$ = 0.69314 71805 59945 30941 72321 21458 17656 80755 00134 36025
$\log_{10} 2$ = 0.30102 99956 63981 19521 37388 94724 49302 67681 89881 46210
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$\log_{10} 3$ = 0.47712 12547 19662 43729 50279 03255 11530 92001 28864 19069

OTHER CONSTANTS

Euler's Constant γ = 0.57721 56649 01532 86061
$\log_e \gamma$ = −0.54953 93129 81644 82234
Golden Ratio ϕ = 1.61803 39887 49894 84820 45868 34365 63811 77203 09180

The Golden Ratio

6 *Numerical Constants*

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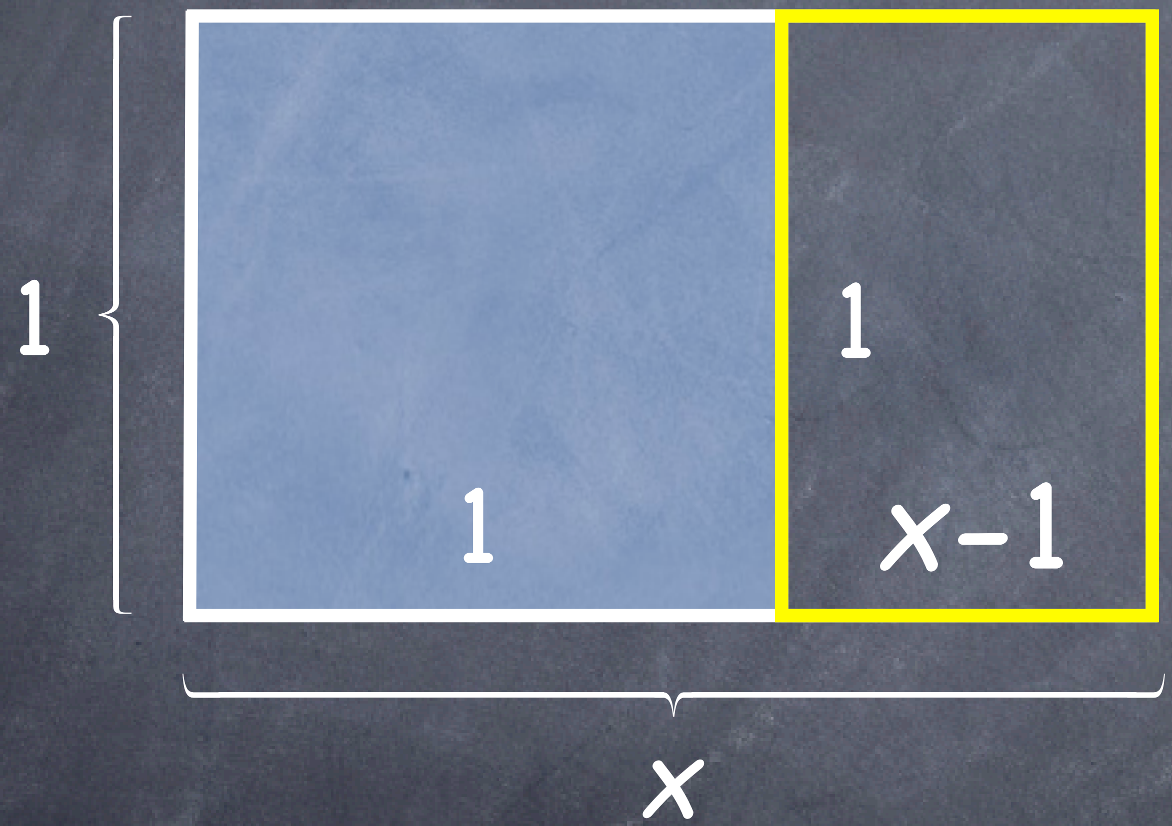
The Golden Rectangle:

A rectangle with the property that the removal of a square results in a new rectangle that has the same proportions as the original.



$$\frac{x}{1} = \frac{1}{x-1}$$

$$x^2 - x = 1$$

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} \approx 1.618\ldots = \phi$$

A Small Problem

Theorem: If k is not a perfect square, then $\sqrt{k} \notin \mathbb{Q}$.

The Golden Ratio: $\phi = \dfrac{1+\sqrt{5}}{2} \notin \mathbb{Q}$

Theorem: If k is not a perfect nth power, then $\sqrt[n]{k} \notin \mathbb{Q}$.

A Famous Irrational – e

Consider the expression: $(1+1/n)^n$

n	$(1+1/n)^n$
1	2
10	2.59374
100	2.70481
1000	2.71692
10000	2.71815

$$\lim_{n\to\infty}(1+1/n)^n = e \approx 2.718281828459045\ldots$$

A Famous Irrational – e

Proved the irrationality of e and e^2 in 1737.

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \ddots}}}}}}$$

Leonard Euler
1707–1783

A Proof of the Irrationality of e

Assume that $e = \frac{N}{D}$, where N and D have no common factor.

Recall: $$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Then we have:

$$\frac{N}{D} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{D!} + \sum_{n=D+1}^{\infty} \frac{1}{n!}$$

A Proof of the Irrationality of e

$$\frac{N}{D} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{D!} + \sum_{n=D+1}^{\infty} \frac{1}{n!}$$

Multiply both sides by $D!$ to get

$$N(D-1)! = D! + \frac{D!}{1!} + \frac{D!}{2!} + \frac{D!}{3!} + \cdots + \frac{D!}{D!} + \sum_{n=D+1}^{\infty} \frac{D!}{n!}$$

Note that $N(D-1)!$ is an integer, as are the terms

before $\sum_{n=D+1}^{\infty} \frac{D!}{n!}$. Thus, $\sum_{n=D+1}^{\infty} \frac{D!}{n!}$ is an integer.

A Proof of the Irrationality of e

But,

$$\sum_{n=D+1}^{\infty} \frac{D!}{n!} = \frac{1}{D+1} + \frac{1}{(D+1)(D+2)} + \frac{1}{(D+1)(D+2)(D+3)} + \cdots$$

$$< \frac{1}{D+1} + \frac{1}{(D+1)^2} + \frac{1}{(D+1)^3} + \cdots$$

This last sum is a geometric series and

$$\frac{1}{D+1} + \frac{1}{(D+1)^2} + \frac{1}{(D+1)^3} + \cdots = \frac{\frac{1}{D+1}}{1 - \frac{1}{D+1}} = \frac{1}{D}.$$

A Proof of the Irrationality of e

This means that $0 < \sum_{n=D+1}^{\infty} \frac{D!}{n!} < \frac{1}{D}$.

Thus $\sum_{n=D+1}^{\infty} \frac{D!}{n!}$ cannot be an integer, as shown earlier. Contradiction! Hence, $e \notin \mathbb{Q}$.

Also, $\sin(1/n)$, $\cos(1/n)$, and $e^{1/n} \notin \mathbb{Q}$ for every positive integer n.

Another Famous Irrational – π

Showed that:

If x is a rational number other than zero, the value of $\tan(x)$ is irrational.

Since $\tan(\pi/4) = 1$, it follows that $\pi/4$ and hence π is irrational.

Johann Lambert
1728–1777

Another Famous Irrational – π

Showed that:

Johann Lambert
1728–1777

If x is a rational number other than zero, the value of $\tan(x)$ is irrational.

This result was extended to include the irrationality of $\sin x$, $\cos x$, and e^x for all rational $x \neq 0$.

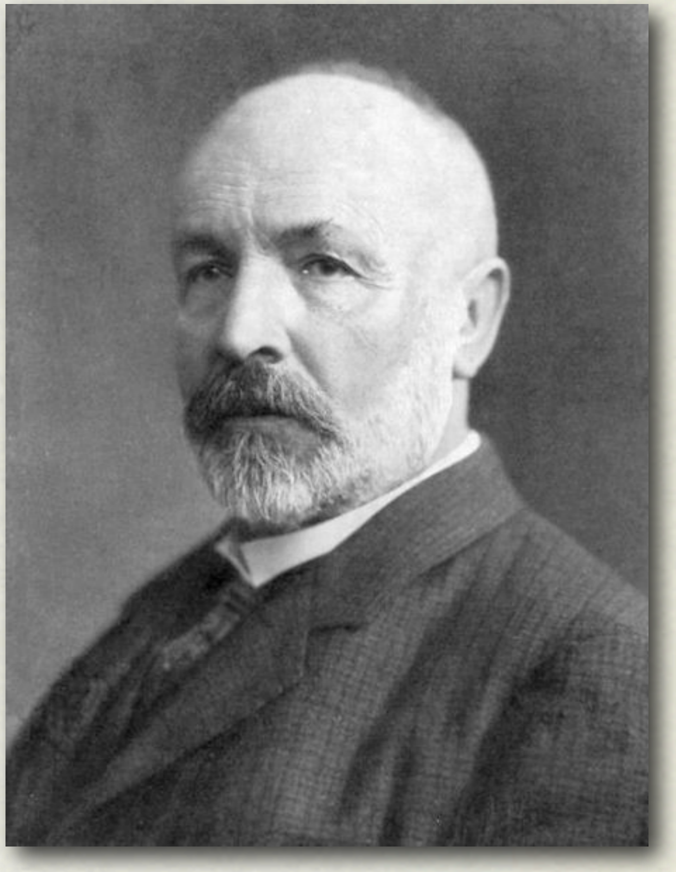
Dr. Orval Klose

An Extraordinary Statement

"It may surprise you to learn that the set of irrationals is more numerous than the set of rationals."

The Infinities of Georg Cantor

Georg Cantor
1845–1918



In 1874 Cantor published an article, titled "*On a Property of the Collection of All Real Algebraic Numbers*."

This article was the first to provide a rigorous proof that there was more than one kind of infinity.

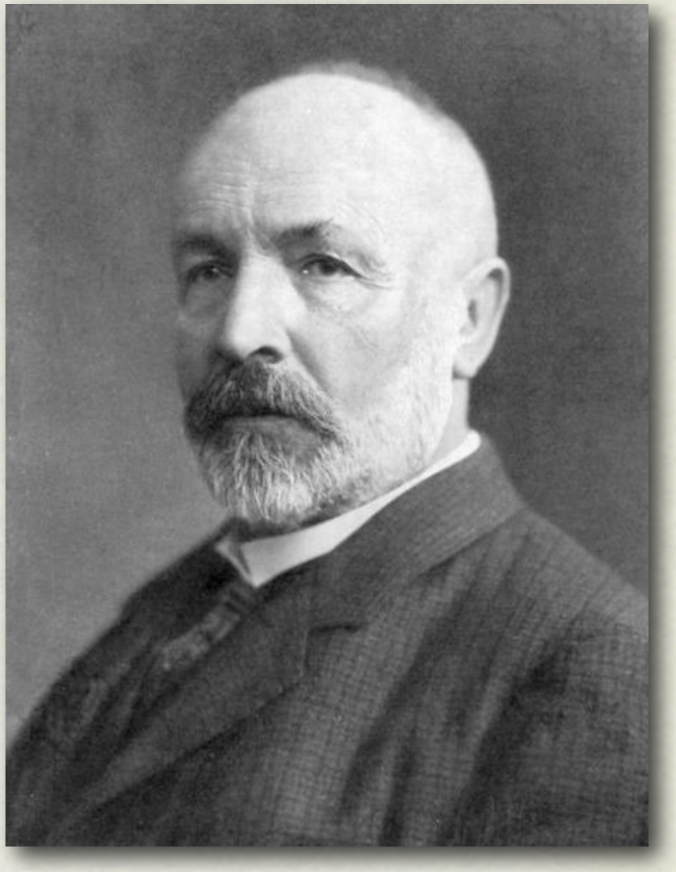
The Infinities of Georg Cantor

Set: A collection of objects.

$$\{a,\ b,\ c,\ \ldots,\ z\}$$

$$\{1,\ 2,\ 3,\ \ldots\}$$

Georg Cantor
1845–1918



Cardinality: The number of elements in a set.

Notation: $n(A)$

Example: $n(\{a,\ b,\ c,\ \ldots,\ z\}) = 26$

The Infinities of Georg Cantor

One-to-one correspondence:

A rule that assigns to each element of one set, one and only one element of a second set, with no element omitted.

$$\begin{matrix} \{1, & 2, & 3, & 4, & 5\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{a, & e, & i, & o, & u\} \end{matrix}$$

The Infinities of Georg Cantor

One-to-one correspondence:

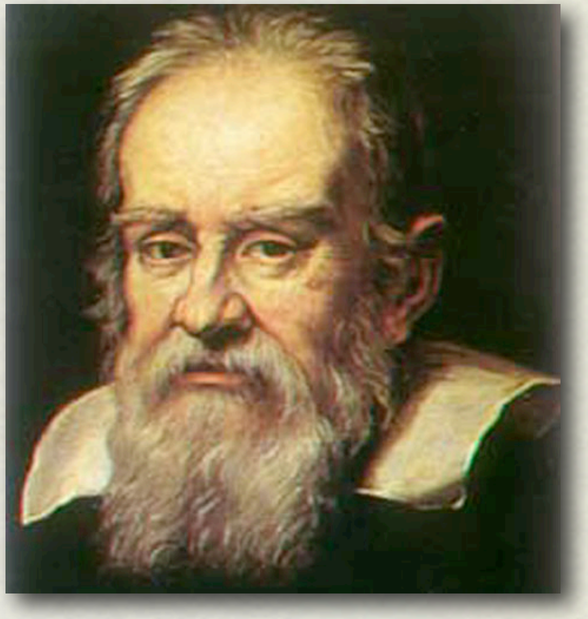
A rule that assigns to each element of one set, one and only one element of a second set, with no element omitted.

$$\begin{array}{cccccc} \{1, & 2, & 3, & 4, & 5, & \ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \{2, & 4, & 6, & 8, & 10, & \ldots\} \end{array}$$

The Infinities of Georg Cantor

Discourses Concerning the Two New Sciences (1638)

Galileo Galilei
1564–1642



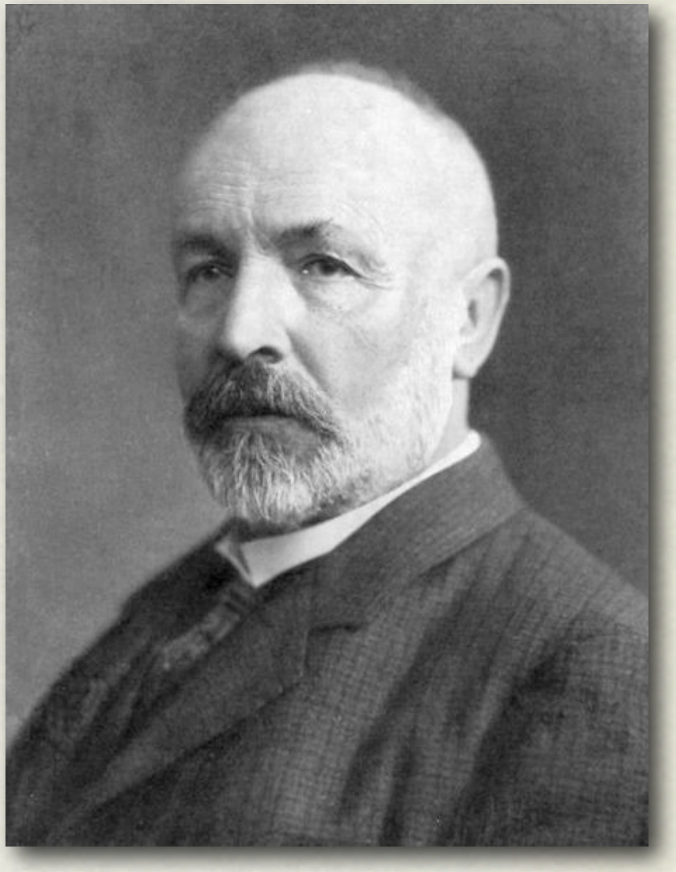
$$\begin{array}{ccccc} \{1, & 2, & 3, & 4, & 5, \ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{1, & 4, & 9, & 16, & 25, \ldots\} \end{array}$$

"So far as I see, we can only infer that the number of squares is infinite and the number of their roots is infinite."

The Infinities of Georg Cantor

Postulate:

Georg Cantor
1845–1918



Whenever two sets – finite or infinite – can be matched by a one-to-one correspondence, they have the same number of elements.

$$
\begin{aligned}
n(\{1, 2, 3, \ldots\}) &= n(\{2, 4, 6, \ldots\}) \\
&= n(\{1, 4, 9, \ldots\}) \\
&= n(\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\})
\end{aligned}
$$

The Infinities of Georg Cantor

Denumerable:

Any set that can be placed into a one-to-one correspondence with the natural numbers.

Examples:

The even numbers,

the squares,

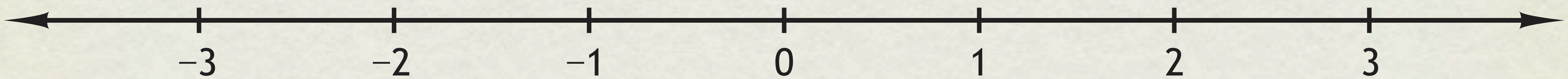
the integers,

the primes and the rationals!

The Infinities of Georg Cantor

Notation: $n(\mathbb{N}) = \aleph_0$ (aleph-null)

Thus, $n(\mathbb{N}) = n(\mathbb{Z}) = n(\mathbb{Q}) = \aleph_0$



The real number line: $\mathbb{R}$

Cantor showed: $n(\mathbb{N}) < n(\mathbb{R}) = c$ (continuum)

The Infinities of Georg Cantor

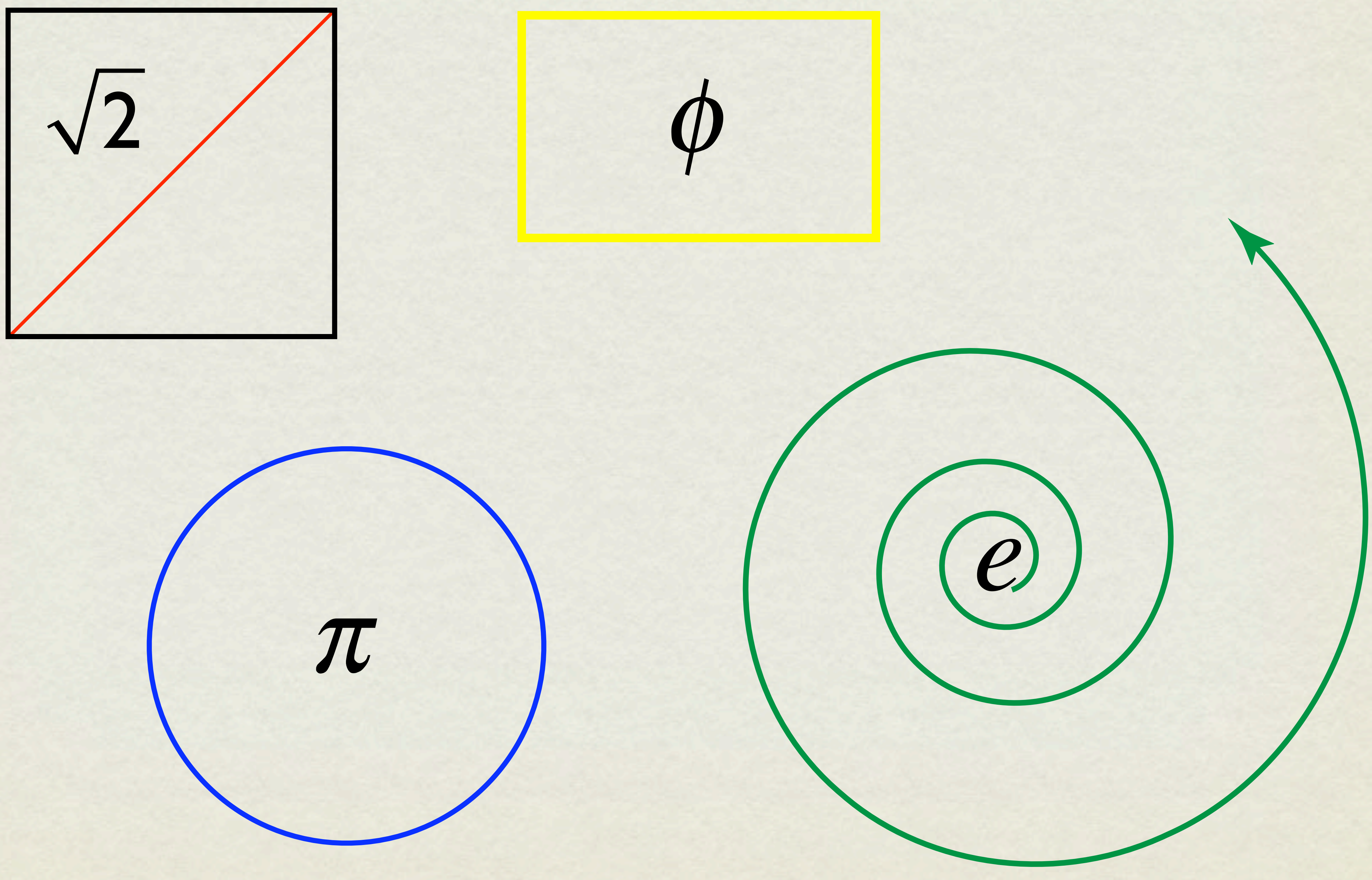
Now, $\text{Reals} = \text{Rationals} \cup \text{Irrationals}$

and $n(\text{Rationals}) = \aleph_0$.

But $n(\text{Reals}) = c > \aleph_0$, so $n(\text{Irrationals}) > \aleph_0$.

Thus, $n(\text{Irrationals}) > n(\text{Rationals})$.

The Irrational Hall of Fame



Algebraic Numbers

Algebraic: A number that is a solution to a polynomial equation with integer coefficients.

$$\frac{a}{b} \longrightarrow bx - a = 0$$

$$\sqrt{7} \longrightarrow x^2 - 7 = 0$$

$$2 + \sqrt{3} \longrightarrow x^2 - 4x + 1 = 0$$

$$\sqrt[3]{-2 + \sqrt{6}} \longrightarrow x^6 + 4x^3 - 2 = 0$$

$\sqrt{2}$ and ϕ are algebraic

Algebraic Numbers

Algebraic: A number that is a solution to a polynomial equation with integer coefficients.

Are there any **non-algebraic** irrational numbers?

Non-Algebraic Numbers

Transcendental:

Joseph Liouville
1809–1882

An irrational number that is not algebraic.

Liouville's constant:

$$\frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \frac{1}{10^{4!}} + \cdots$$

$$= 0.110001000000000000000001000\ldots$$

Transcendental Numbers

Transcendental: An irrational number that is not algebraic.

Charles Hermite
1822–1901

e is transcendental

"I shall risk nothing on an attempt to prove the transcendence of π. If others undertake this enterprise, no one will be happier than I in their success. But believe me, it will not fail to cost them some effort."

Transcendental Numbers

Transcendental: An irrational number that is not algebraic.

Charles Hermite
1822–1901

e is transcendental

Ferdinand von Lindemann
1852–1939

π is transcendental

The Infinities of Georg Cantor

Reals $(\mathbb{R})$ $=$ Algebraic $(\mathbb{R}_A)$ $\cup$ Transcendentals

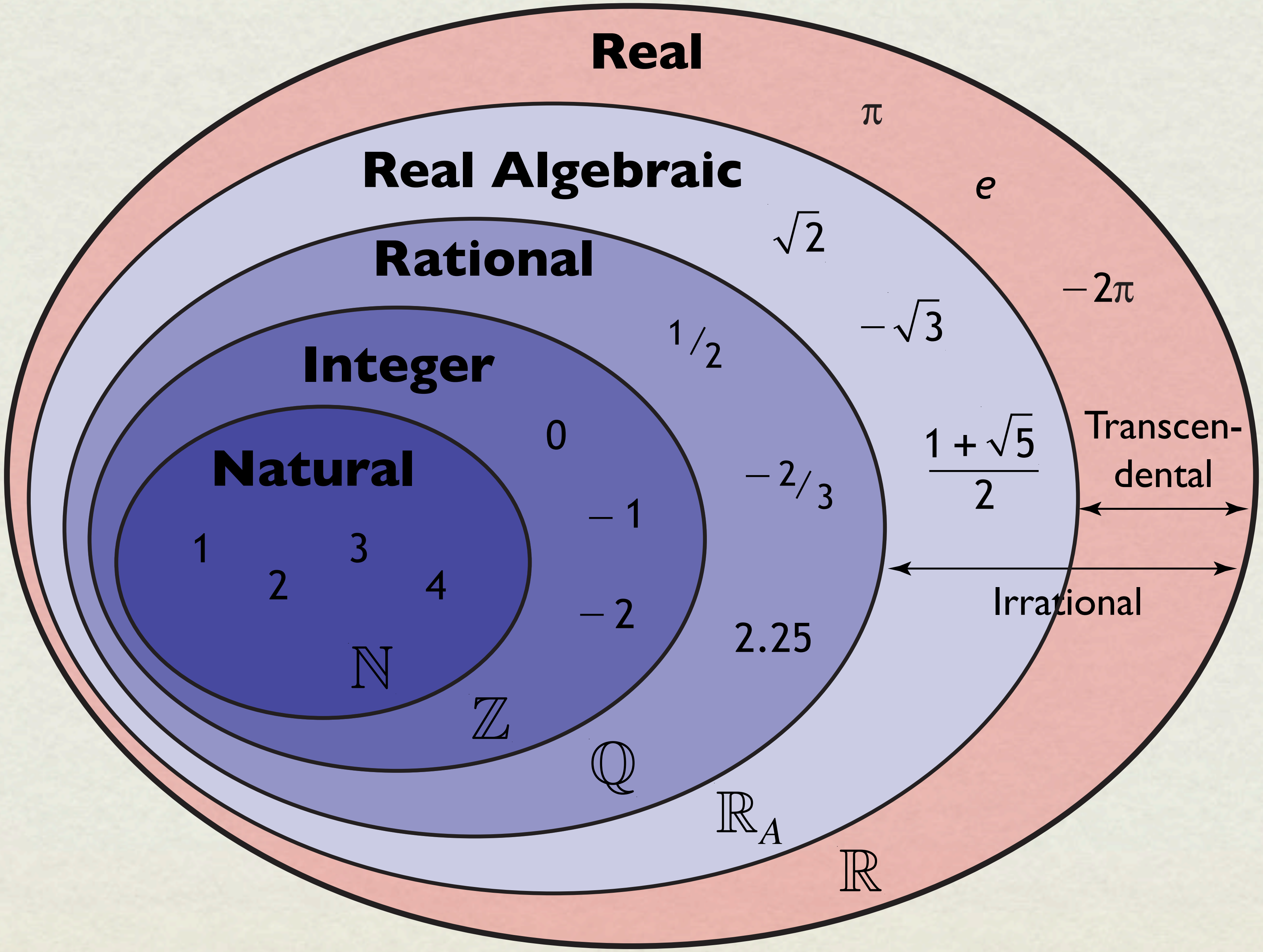
What about $n(\mathbb{R}_A)$ and $n(\text{Transcendentals})$?

In 1874 Cantor showed that $n(\mathbb{R}_A) = \aleph_0$.

Hence, $n(\text{Transcendentals}) > \aleph_0$.

Thus, **most** real numbers are irrational and **most** irrational numbers are transcendental!

The Real Number System



The Property of Closure

The sum of any two natural numbers is another natural number.

The naturals are **closed** under addition.

The integers are **closed** under subtraction.

The Property of Closure

Rational	Irrational
$\sqrt{2}-\sqrt{2} = 0$	$\sqrt{3}+\sqrt{3} = 2\sqrt{3}$
$\sqrt{3}\cdot\sqrt{12} = 6$	$\sqrt{7}\cdot\sqrt{3} = \sqrt{21}$
$\dfrac{\sqrt{24}}{\sqrt{6}} = 2$	$\dfrac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$

The set of irrationals is not closed under the operations of addition, subtraction, multiplication and division.

The Property of Closure

The set of irrationals is not closed under the operations of addition, subtraction, multiplication and division.

What about exponentiation? a^b

If *a* and *b* are rational, then a^b may be either rational $9^{1/2} = 3$ or irrational $2^{1/2} = \sqrt{2}$.

The rationals are not closed under exponentiation.

The Property of Closure

If a and b are rational, then a^b may be either rational or irrational.

The same is true if a and b are irrational.

Observation #1: An irrational number to an irrational power may be rational.

The Property of Closure

Observation #1: An irrational number to an irrational power may be rational.

To show this, we need an example a^b where a and b are irrational and a^b is rational.

If $\sqrt{2}^{\sqrt{2}}$ is rational, then it is our example.

If $\sqrt{2}^{\sqrt{2}}$ is irrational, then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$ is our example. Q.E.D.

The Property of Closure

Observation #2: An irrational number to an irrational power may be irrational.

To show this, we need an example a^b where a and b are irrational and a^b is irrational.

If $\sqrt{2}^{\sqrt{2}}$ is irrational, then it is our example.

If $\sqrt{2}^{\sqrt{2}}$ is rational, then $\sqrt{2}^{\sqrt{2}+1} = \sqrt{2}^{\sqrt{2}}\sqrt{2}$ is our example. Q.E.D.

The Property of Closure

Is $\sqrt{2}^{\sqrt{2}}$ rational or irrational?

In 1930, Rodion Kuzmin proved that $2^{\sqrt{2}}$ is a transcendental number.

But $\sqrt{2}^{\sqrt{2}} = \sqrt{2^{\sqrt{2}}}$, so $\sqrt{2}^{\sqrt{2}}$ is irrational.

Algebraic or Transcendental?

Conjecture:

If a and b are algebraic numbers with a not equal to 0 or 1, and if b is not a rational number, then the number a^b is transcendental.

David Hilbert
1862–1943

Proved by Aleksandr Gelfand and Theodor Schneider, independently, in 1934.

Algebraic or Transcendental?

Gelfand-Schneider theorem

If a and b are algebraic numbers with a not equal to 0 or 1, and if b is not a rational number, then the number a^b is transcendental.

From this it follows that $2^{\sqrt{2}}$ and $\sqrt{2}^{\sqrt{2}}$ are transcendental.

Also that e^{π} is transcendental.

Algebraic or Transcendental?

Gelfand-Schneider theorem

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From this it follows that $2^{\sqrt{2}}$ and $\sqrt{2}^{\sqrt{2}}$ are transcendental.

Also that $e^{\pi} = \left(e^{i\pi}\right)^{-i} = (-1)^{-i}$ is transcendental.

The classifications of π^{π}, π^{e}, and e^{e} are unknown.

Final Thoughts

Final Thoughts

Edward Titchmarsh
1888–1963

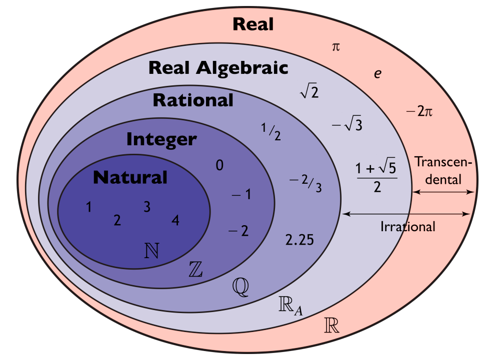
"It can be of no practical use to know that Pi is irrational, but if we can know, it surely would be intolerable not to know."

SOME IRRATIONALS I HAVE KNOWN

John Martin

Santa Rosa Junior College

Some Irrationals I Have Known



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My top ten favorite irrationals:

1. Pythagoras's Constant $\sqrt{2}$
2. The Golden Ratio ϕ
3. Archimedes's Constant π
4. The Base of the Natural Logarithm e
5. Liouville's Number $0.110001000000000000000001000\ldots$
6. Hilbert's Number $2^{\sqrt{2}}$
7. Gelfond's Constant e^{π}
8. $i^i = e^{-\pi/2}$
9. Apéry's constant $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$
10. Champernowne's number $0.123456789101112131415\ldots$

Additional Topics to Explore:

Gelfond-Schneider theorem

Transfinite Cardinals

Slides Used in the Presentation:

http://online.santarosa.edu/homepage/jmartin/

Scroll to the bottom for a link to a folder containing a PDF of the slides.
