Problems to Open the Math Appetite of Non-Mathematicians

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A simple probability problem

Problem

A businessman traveling to Europe writes three letters: to his boss, to his mistress, and to his wife. When putting the letters in the envelopes he's distracted and puts them randomly. What is the probability he'll be in trouble?

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Solution : Let us denote by B the letter sent to the boss, M the letter sent to the mistress, and W the letter sent to the wife. Writing a sequence like WBM we understand that the boss got the letter for the wife, the mistress to one sent to the boss, and the wife the letter sent to the mistress. All the possibilities are then

$$\text{BMW, BWM, MWB, MBW, WMB, WBM}$$

and the only one with no troubles is the BMW.

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Solution : As usual this tenis tournaments will be by elimination. At each match one player is eliminated. To establish the winner, 126 players have to eliminated, therefore there will be 126 matches.

Geometry on a sphere. Antipode

Problem

On the Buenos Aires airport the owner of a jet prepares for the take off when a misterious woman asks him for help. "My grandfather is very ill and I have to get to see him before it is too late. There is no seat on any flight to my home country for the next 3 days." "Don't worry Miss. I'll drop you there. It is in my way" said the gentleman touched by her story. "But I did not tell you where I'm going!" said the woman suprised.
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Solution : The shortest path between two points on a sphere is on a big diameter. Given two points A and B on a sphere, we can find a diameter passing through them and any other point C, if and only if A and B are antipodal.

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The antipodal point of Buenos Aires is Shanghai in China.

The diagonal is bigger than the sides

Problem

Professor Joe Metri is flying to Lake Tahoe for a ski vacation. At the airport he discovers that the airline is not accepting anymore objects with one of the dimensions greater than 4 feet. The skis were 5 feet long and he could not leave without them, so he tried to persuade the airline representatives to be allowed to carry them. The only thing he obtained was a seat on another flight a few hours later.

Having no other choice Professor Metri accepted and later in the day he returned to the airport and was accepted with his skis. How did he proceed?

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Solution : The diagonal of a cube with the side 4 feet is $4\sqrt{3}$ which is greater than 5 feet.

Cut the cake

Problem

Mama Bear prepared for her two bear cubs a delicious chocolate cake that has a rectangular shape. To test her geometry skills, Papa Bear cuts and eats a rectangular region somewhere in the middle of the cake. How can Mama Bear make two pieces of equal surface with one straight line cut?

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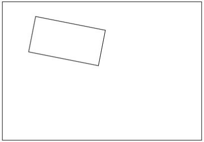

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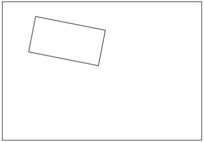

Solution : Any straight line passing through the center of a rectangle is dividing it into two regions of equal areas.

Greatest common divisor

Problem

Having only a mold of a 17° *angle, draw a* 53° *angle.*

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Solution : Since the greatest common divisor of 17 and 360 is 1, drawing 360 angles of 17° around a point we divide the 360° around the point into 360 angles of 1°. We take 53 of them.

Algebraic identity

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Solution : The identity

$$(X+7)^2+(X+4)^2+(X+2)^2+(X+1)^2 = (X+6)^2+(X+5)^2+(X+3)^2+X^2$$

shows we can divide any 8 consecutive perfect squares in two sets with the same sum. Since 2008 is divisible by 8 the answer is yes.

One-to-one functions

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A legend gives credit for the idea of this problem to late Grigore Moisil. In the middle of a discussion at the Romanian Academy, being provoked with the eternal question "Where do we use (so much) Mathematics?", Moisil proposed his fellow academicians the above problem. In his original version the gloves were replaced by condoms.

Arrow's theorem

Problem

A triathlon consists in swimming, cycling, and running over various distances. Anna, Bianca and Carla participate at a triathlon. At the end of the competition the conclusion of their coach is that Anna did better than Bianca, Bianca did better than Carla and Carla did better than Anna. Is this possible?

The coach considers that "A did better than B" for the entire triathlon if A did better than B in more components than B did better than A.

Solution : The order in the three components is

swimming A, B, C
cycling B, C, A
running C, A, B

Numbers base 2

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Solution : We take 1 coin from the first bag, 2 from the second, 2^2 from the third, ..., 2^7 from the eight bag, with a total of $2^8 - 1 = 255$ coins. We weigh them and then we subtract the weight from 2550. We write the result in base 2 and the position of the digits 1 shows the bags with 9 grams bags.

Numbers base 2 and base 3 I

Problem

The inquisitive customer looked at the 5 different weights standing in a line beside the balance scale on farmer Ben's counter in the Farmers Market. "Are those 5 weights enough", he asked, "to measure out any weights your customers ask for?"
"By putting one or more of these weights on one pan", Ben replied, "I can weigh on the other pan any whole number of pounds from 1 to 31."
His neighbor, Tom, overhearing them, broke into the conversation. "I do better than that," he said. "I have only four weights. But, by putting weights on both pans of my scale, I can weigh out any whole number of pounds from 1 to 40."
a) What are the five weights that Ben uses?
b) What are the four weights that Tom uses?

Numbers base 2 and base 3 II

Solution : a) (1,2,4,8,16)
b) (1,3,9,27)

Broken calculator

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We have a scientific calculator on which the keys 0,1,3,4,5,6,7,8,9 (all digits except 2) are not working. Can we make the calculator show any integer number?

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$$-n = \log_2\left(\log_2\left(\underbrace{\sqrt{\sqrt{\ldots\sqrt{2}}}}_{n-radicals}\right)\right)$$

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It seems this problem was created by John von Neumann while attending a presentation he did not enjoy.

The invariance principle. Sum of series I

Problem

Three aliens are caught in the red prison (see figure 1). Each one of them can escape from its cell transforming himself in two clones, one in the cell above and one in cell on the right, but only if these are empty. Such transformation is ilustrated in the fiigure 2. The same rule applies also to the clones. Can all three aliens escape from the prison in finitely many moves?

The invariance principle. Sum of series II

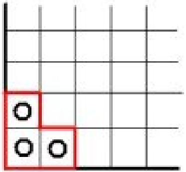

fig.1

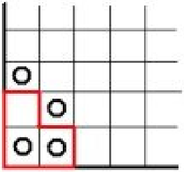

fig.2

Solution : We label the cells by pairs of non-negative integers

$$(x, y)\,, \quad x, y \in \mathbb{N}\,.$$

At every moment, if M is the set of cells occupied by the aliens and their clones, denote

$$S = \sum_{(x,y)\in M} \frac{1}{2^{x+y}}\,.$$

The invariance principle. Sum of series III

Since

$$\frac{1}{2^{x+1+y}} + \frac{1}{2^{x+y+1}} = \frac{1}{2^{x+y}}\,, \quad \forall x, y$$

the sum S is invariant to transformations. For the initial configuration

$$S = \frac{1}{2^{0+0}} + \frac{1}{2^{0+1}} + \frac{1}{2^{1+0}} = 2\,.$$

while if M is any finite set of cells outside the red prison

$$\sum_{(x,y)\in M} \frac{1}{2^{x+y}} < \sum_{n\geq 2} \frac{n+1}{2^n} = 2\,.$$

Parity problem I

Problem

31 people decide to play rugby (two teams of 15 players and a ref). To have a fair game, they agree that the total weight of players to be the same for the two teams. Checking their weights they discover that for each choice of the ref, there is a partition of the other 30 players in two fair teams. Prove that all the players have the same weight.

Solution : Let $M = \{w_1, w_2, ..., w_{31}\}$ be the set of weights. This set has the property

P: the set obtained after removal of any element can be partitioned into two subsets of 15 elements of equal sum.

Let us observe that the elements of M have the same parity. For this let $S = \sum_{k=1}^{31} w_k$. Since $\{w_2, w_3, ..., w_{31}\}$ can be partitioned into

Parity problem II

two subsets with the same sum, $S - w_1$ is even. Then w_1 has the same parity with S, and similarly with the others.
Assuming $w_1 \leq w_2 \leq ... \leq w_{31}$ let $x_i = w_i - w_1$. The set $X = \{x_1, x_2, ..., x_{31}\}$ has also property P, therefore its elements have the same parity with $x_1 = 0$ that is they're even. Let $y_i = x_i/2$, for all $i = \overline{1, 31}$. The set $Y = \{y_1 = 0, y_2, ..., y_{31}\}$ has also property P. Using the method of infinite descent we obtain $x_1 = x_2 = ... = x_{31} = 0$ and $w_1 = w_2 = ... = w_{31}$.

Hopper configurations

Problem

Ten hoppers sit on a circle. At a given moment all start moving on the circle with the same constant speed in a direction of their choice. When two hoppers meet they both turn around and continue moving in the opposite direction. Prove that at some moment the hoppers will be in the same configuration as the initial one.

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Solution : Let's imagine that each hopper wears a hat, and when they meet they exchange hats. Then the hats moves with constant speed on the circle.

Time measurement

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We have two candles that are burning with the same speed over their entire length for exactly one hour. How can we measure 15 minutes using only these candles?

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We have two candles that are burning with the same speed over their entire length for exactly one hour. How can we measure 15 minutes using only these candles?

Solution : We light both candles but one of them on both sides. This one will burn completely in $1/2$ hour. At that moment we extinguish the second candle. If we light this half candle on both sides it will burn forr exactly 15 minutes.

Marbles

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You are given 2 marbles and the task of determining the highest story of a 100 story building from which you can drop a marble without breaking it. Find a scheme that is guaranteed to give the answer in the fewest number of drops.

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Solution : With only one marble the strategy would be to throw it succesively from each floor. With two marbles, if we throw the first marble from the 12th floor, then 12 is the best number we can hope (if the first breaks then we need to use the other marble succesively on each floor). Also, if the first marble is not crushed from the 12th floor to keep the maximum number of trials at 12 we will throw it again at the floor 12+11=23, and then 12+11+10, etc. We get to solve the inequation

$$n + (n-1) + (n-2) + ... + 1 \geq 100$$

with the minimal solution 14

Coffee or cream

Problem

You are given a cup of coffee and a cup of cream, each containing the same amount of liquid. A spoonful of cream is taken from the cup and put into the coffee cup, then a spoonful of the mixture is put back into the cream cup. Is there now more or less cream in the coffee cup than coffee in the cream cup?

Tilings

Problem

A $8' \times 8'$ *bathroom is to have its floor tiled. Each tile is* $2' \times 1'$. *In one corner of the bathroom is a sink, and its plumbing occupies a* $1' \times 1'$ *square in the floor. In the opposite corner is a toilet, and its plumbing occupies a* $1' \times 1'$ *square in the floor. Is it possible to achieve the required tiling of the floor?*